Stabilized Galerkin and Collocation Meshfree Methods

J. S. Chen[1], **S. W. Chi**[1], **and H. Y. Hu**[2]

[1] *Department of Civil & Environmental Engineering, University of California, Los Angeles*
Los Angeles, CA 90095, USA
e-mail: jschen@seas.ucla.edu, chishengwei@ucla.edu

[2] *Department of Mathematics, Tunghai University*
Taichung 407, Taiwan R.O.C.
e-mail: huhy@math.thu.edu.tw

Abstract

Meshfree methods have been formulated based on Galerkin type weak formulation and collocation type strong formulation. The approximation functions commonly used in the Galerkin based meshfree methods are the moving least-squares (MLS) and reproducing kernel (RK) approximations, while the radial basis functions (RBFs) are usually employed in the strong form collocation method. Galerkin type formulation in conjunction with approximation functions with polynomial reproducibility yields algebraic convergence. Alternatively, strong form collocation method with RBF approximation offers exponential convergence, however the method is suffered from ill-conditioning due to its "nonlocal" approximation. In this work, we discuss stability issues related to nodal integration of Galerkin type meshfree method and ill-conditioning of the radial basis collocation method. We show how to combine the advantages of RBF and RK approximations to yield a local approximation that is better conditioned than that of the radial basis collocation method, while at the same time offers a higher rate of convergence that that of Galerkin type reproducing kernel method.

Keywords: meshless methods, Galerkin method, fracture, composites

1. Introduction

In the past 15 years, meshfree methods have emerged into a new class of computational methods that have been applied to many engineering and scientific problems. Meshfree methods all share a common feature: the approximation of unknown in the partial differential equation is constructed based on scattered points without mesh connectivity. While no mesh is needed in the construction of approximation in meshfree methods, domain integration presents some difficulties if the discrete equation is formulated based on weak formulation, such as the reproducing kernel particle method (RKPM) [1]. Employing the conventional Gauss quadrature rules in Galerkin meshfree method does not yield a solution that passes patch tests for boundary value problems with random point discretization. In the first part of this work, various methods proposed for domain integration of Galerkin meshfree methods will be reviewed, and issues associated with these methods will be addressed. The concept of integration constraint in Galerkin meshfree methods for solving 2nd order PDE will be introduced, and a stabilized conforming nodal integration (SCNI) [2] with gradient smoothing that satisfies first order integration constraint (passing linear patch test) and suppresses zero energy modes of the direct nodal integration will be presented first. The possibility of enhancing SCNI for higher order accuracy will also be discussed [7]. The recent study shows that SCNI eliminates rank instability in the direct nodal integration but exhibits low frequency modes that could be excited under certain conditions [8,9]. Modified SCNI with enhanced stability will be presented. The extension of nodally integrated Galerkin meshfree methods to plates, shells, and large deformation problems have been demonstrated in the literatures [3,4,5,6].

Alternatively, collocation on strong forms has been introduced in meshfree method, such as the radial basis collocation methods (RBCM) [10,11,12] and the reproducing kernel collocation method (RKCM) [13,14]. From convergence standpoint, the compactly supported reproducing kernel approximations with monomial reproducibility render an algebraic convergence in RKCM, while the nonlocal RBFs with certain regularity offer exponential convergence in RBCM. Nevertheless, the linear system of RBCM is typically more ill-conditioned compared to those based on compactly supported approximations. We show how one can localize RBF using a partition of unity paradigm, such as the reproducing kernel enhanced radial basis function [11], to yield a local approximation while maintaining the exponential convergence of RBCM. This localized RBF, combined with the subdomain collocation method, have been applied to problems with local features, such as problems with heterogeneity or cracks that are traditionally difficult to be solved by RBCM [11,12].

2. Stabilized Galerkin Meshfree Method

Domain integration of weak form poses considerable complexity in Galerkin meshfree method. Employment of Gauss quadrature rules yields integration error when background grids do not coincide with the covers of shape functions. Direct nodal integration, on the other hand, results in rank deficiency. Both methods do not pass linear patch test for non uniform point distribution. A stabilization conforming nodal integration (SCNI) [2] has been introduced to meet linear patch test and to remedy rank deficiency of direct nodal integration. For demonstration, consider here a Poisson problem

$$\nabla^2 u + Q = 0 \quad in\ \Omega \tag{1}$$

The corresponding Galerkin approximation is to find $u^h \in H_0^1$, $\forall v^h \in H_0^1$,

$$\int_\Omega \nabla v^h \cdot \nabla u^h d\Omega = \int_\Omega v^h Q d\Omega \tag{2}$$

In SCNI, the gradient evaluated at the nodal point $\mathbf{x}_L$ is calculated as

$$\bar{\nabla} u^h(\mathbf{x}_L) = \frac{1}{w_L}\int_{\Omega_L} \nabla u^h d\Omega = \frac{1}{w_L}\int_{\partial\Omega_L} u^h \mathbf{n} d\Gamma, \quad w_L = \int_{\Omega_L} d\Omega \tag{3}$$

Here Ω_L is the nodal representative domain, which can be obtained from triangulation or Voronoi cell of a set of discrete points as shown in Fig. 1.

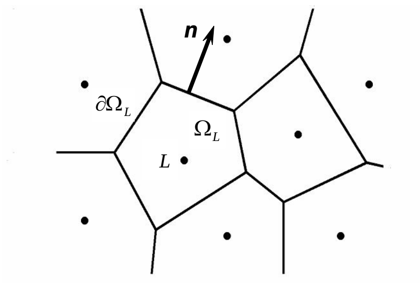

Figure 1: Nodal representative domain

Introducing reproducing kernel (RK) approximation of u, $u^h = \sum_{I=1}^{N} \Psi_I d_I$, with Ψ_I the RK shape functions, into (3), we have

$$\bar{\nabla} u^h(\mathbf{x}_L) = \sum_{I=1}^{N} \bar{\mathbf{B}}_I(\mathbf{x}_L)\mathbf{d}_I \tag{4}$$

where

$$\bar{\mathbf{B}}_I(\mathbf{x}_L) = \frac{1}{w_L}\int_{\partial\Omega_L} \mathbf{n}\Psi_I d\Gamma \tag{5}$$

Introducing the smoothed gradient of (3) into the nodally integrated weak form yields the following discrete equation:

$$\sum_{L=1}^{N} \bar{\nabla} v^h(\mathbf{x}_L)\cdot\bar{\nabla} u^h(\mathbf{x}_L) w_L = \sum_{L=1}^{N} v^h(\mathbf{x}_L) Q(\mathbf{x}_L) w_L \tag{6}$$

A boundary value problem in Fig. 2 is solved by RKPM with direct nodal integration, 5-point Gauss quadrature rule, and SCNI. A much enhanced stability and convergence in SCNI compared to a direct nodal integration is observed. In fact, SCNI results are slightly better than that obtained by 5th order Gauss quadrature.

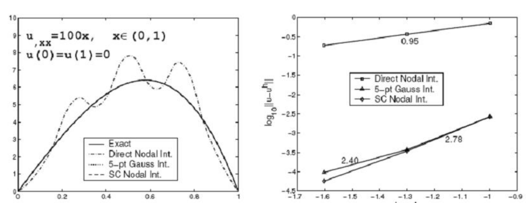

Figure 2: Comparison of RKPM solution using various integration methods.

It has been shown that SCNI for Galerkin weak form passes linear patch test, and it eliminates zero energy modes resulting from the direct nodal integration of Galerkin weak form [2,3]. Although the discrete equation (6) passes a linear patch test, SCNI only offers linear exactness in the Galerkin approximation. To achieve a higher order accuracy, a correction to SCNI can be shown as [7]:

$$\int_\Omega \nabla v^h(\mathbf{x})\cdot\nabla u^h(\mathbf{x}) d\Omega = \sum_{L=1}^{N} \bar{\nabla} v^h(\mathbf{x}_L)\cdot\bar{\nabla} u^h(\mathbf{x}_L) w_L + \underbrace{\sum_{L=1}^{N}\int_{\Omega_L}\left(\nabla v^h(\mathbf{x}) - \bar{\nabla} v^h(\mathbf{x}_L)\right)\cdot\left(\nabla u^h(\mathbf{x}) - \bar{\nabla} u^h(\mathbf{x}_L)\right) d\Omega}_{correction\ term} \tag{7}$$

This approach has also been used for suppressing nonzero energy modes in SCNI, called the modified SCNI (M-SCNI), using a variant of (7) as:

$$\int_\Omega \nabla v^h(\mathbf{x})\cdot\nabla u^h(\mathbf{x}) d\Omega \approx \sum_{L=1}^{N}\Big\{\bar{\nabla} v^h(\mathbf{x}_L)\cdot\bar{\nabla} u^h(\mathbf{x}_L) w_L + \sum_{C\in T_L}\alpha\left(\hat{\nabla} v^h(\mathbf{x}_C) - \bar{\nabla} v^h(\mathbf{x}_L)\right)\cdot\left(\hat{\nabla} u^h(\mathbf{x}_C) - \bar{\nabla} u^h(\mathbf{x}_L)\right) w_C\Big\} \tag{8}$$

where T_L is a set of subcells associated with the Voronoi call of node L, w_C is the corresponding area (or volume) of each subcell, and α is the stabilization parameter. For elasticity, a similar approach is shown in [9]. This approach can be easily applied to large deformation problems, where the nodally smoothed deformation gradient is computed using Eqns. (3)-(4). The first nonzero eigenmodes of the stiffness in 2 dimensional elasticity integrated using SCNI and M-SCNI are compared in Fig. 4.

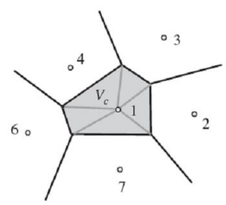

Figure 3: Subcells of Voronio cell

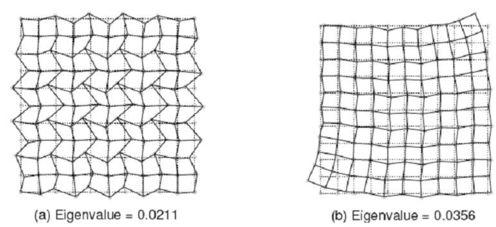

Figure 4: First nonzero eigenmodes of RKPM in 2D elasticity generated using (a) SCNI and (b) M-SCNI

3. Strong Form Collocation Methods

An alternative approach to address domain integration issue in meshfree method is by collocation of strong form. For demonstration, consider a scalar boundary value problem:

$$\begin{aligned} Lu(\mathbf{x}) &= f(\mathbf{x}) \quad \forall \mathbf{x}\in\Omega \\ B^h u(\mathbf{x}) &= h(\mathbf{x}) \quad \forall \mathbf{x}\in\partial\Omega^h \\ B^g u(\mathbf{x}) &= g(\mathbf{x}) \quad \forall \mathbf{x}\in\partial\Omega^g \end{aligned} \tag{9}$$

where Ω is the problem domain, $\partial\Omega^h$ is the Neumann boundary, $\partial\Omega^g$ is the Dirichlet boundary, $\partial\Omega^h \cup \partial\Omega^g = \partial\Omega$, L is the differential operator in Ω, B^h is the differential operator on $\partial\Omega^h$, and B^g is the operator on $\partial\Omega^g$. Introducing approximation of $u^h(\mathbf{x}) = \sum_{I=1}^{N_s} g_I(\mathbf{x}) d_I$ into (8), and enforcing the residuals to be zero at the Nc collocation points $\{\xi_J\}_{J=1}^{N_c} \in \Omega\cup\partial\Omega$, we have

$$\begin{aligned} \sum_{I=1}^{N_s} L g_I(\xi_J) d_I &= f(\xi_J) \quad \forall \xi_J \in\Omega \\ \sum_{I=1}^{N_s} B^h g_I(\xi_J) d_I &= h(\xi_J) \quad \forall \xi_J \in\partial\Omega^h \\ \sum_{I=1}^{N_s} B^g g_I(\xi_J) d_I &= g(\xi_J) \quad \forall \xi_J \in\partial\Omega^g \end{aligned} \tag{10}$$

where $g_I(\mathbf{x})$ is the approximation function. Note that Eqn. (10) is an overdetermined system if $N_c > N_s$, and a least-squares method can be used for the solution. It has been pointed out by Hu and Chen et al. [10] that (10) yields unbalanced errors between domain and boundary collocation equations, and they proposed a weighted collocation method as

$$\begin{aligned} &\sum_{I=1}^{N_s} Lg_I(\xi_J)d_I = f(\xi_J) \quad \forall \xi_J \in \Omega \\ &\sqrt{\alpha^h}\sum_{I=1}^{N_s} B^h g_I(\xi_J)d_I = \sqrt{\alpha^h}h(\xi_J) \quad \forall \xi_J \in \partial\Omega^h \\ &\sqrt{\alpha^g}\sum_{I=1}^{N_s} B^g g_I(\xi_J)d_I = \sqrt{\alpha^g}g(\xi_J) \quad \forall \xi_J \in \partial\Omega^g \end{aligned} \tag{11}$$

where α^h and α^g are the weights for the Neumann boundary $\partial\Omega^h$ and Dirichlet boundary $\partial\Omega^g$, respectively. For balanced errors in the domain and on the boundaries, it was proposed [10] that the weights be selected as $\sqrt{\alpha^h} \approx O(1)$ and $\sqrt{\alpha^g} \approx O(\kappa N_s)$, where κ is the maximum coefficient involved in the differential operator L and the boundary operator B^h. A Poisson problem is solved by a direct collocation method (DCM) in (10) and the weighted direct collocation method (W-DCM) in (11) using RBFs, and the large errors on the boundaries in DCM has been corrected by W-DCM as shown in Fig. 5.

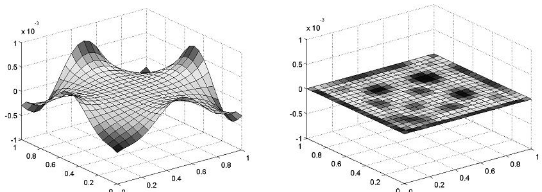

Figure 5: The error distribution of solution obtained using DCM and W-DCM in a Poisson problem

A commonly used approximation function in the strong form collocation method is the radial basis function (RBF), and the approach is called the radial basis collocation method (RBCM). Standard radial basis function offers exponential convergence, however the method is suffered from the large condition numbers due to its "nonlocal" approximation. The reproducing kernel (RK) function, on the other hand, provides polynomial reproducibility in a "local" approximation, and the corresponding discrete system exhibits a relatively small condition number. Nonetheless, reproducing kernel function produces only algebraic convergence. An approach has been proposed to combine the advantages of radial basis function and reproducing kernel function to yield a local approximation that is better conditioned than that of the radial basis function, while at the same time offers a higher rate of convergence than that of reproducing kernel function:

$$u^h(\mathbf{x}) = \sum_{I=1}^{N}\left[\Psi_I(\mathbf{x})\left(a_I + \sum_{J=1}^{M} g_I^J(\mathbf{x})d_I^J\right)\right] \tag{12}$$

where $\Psi_I(\mathbf{x})$ is the RK function with compact support, and $g_I^J(\mathbf{x})$ are RBF's. This approximation, combined with the subdomain collocation method, has been applied to problems with local features, such as problems with heterogeneity (weak discontinuity) or cracks (strong discontinuity). We describe the main ideas in the followings.

Take heterogeneous elasticity as an example as shown in Fig. 6, the following subdomain radial basis collocation method is introduced [11]:

$$\begin{cases} \mathbf{L}^+\mathbf{u}^+ = \mathbf{f}^+ & \text{in } \Omega^+ \\ \mathbf{B}_g^+\mathbf{u}^+ = \mathbf{g}^+ & \text{on } \partial\Omega^+ \cap \partial\Omega_g \\ \mathbf{B}_h^+\mathbf{u}^+ = \mathbf{h}^+ & \text{on } \partial\Omega^+ \cap \partial\Omega_h \end{cases} \tag{13}$$

$$\begin{cases} \mathbf{L}^-\mathbf{u}^- = \mathbf{f}^- & \text{in } \Omega^- \\ \mathbf{B}_g^-\mathbf{u}^- = \mathbf{g}^- & \text{on } \partial\Omega^- \cap \partial\Omega_g \\ \mathbf{B}_h^-\mathbf{u}^- = \mathbf{h}^- & \text{on } \partial\Omega^- \cap \partial\Omega_h \end{cases} \tag{14}$$

$$\begin{cases} \mathbf{u}^+ - \mathbf{u}^- = \mathbf{0} \\ \mathbf{B}_h^+\mathbf{u}^+ + \mathbf{B}_h^-\mathbf{u}^- = \mathbf{0} \end{cases} \text{on } \Gamma, \tag{15}$$

The solution in each subdomain is approximated by separate set of basis functions:

$$u_i^h(\mathbf{x}) = \begin{cases} u_i^{h+}(\mathbf{x}) = g_1^+(\mathbf{x})a_{i1}^+ + \cdots + g_{N_s^+}^+(\mathbf{x})a_{iN_s^+}^+, & \mathbf{x} \in \bar{\Omega}^+ \\ u_i^{h-}(\mathbf{x}) = g_1^-(\mathbf{x})a_{i1}^- + \cdots + g_{N_s^-}^-(\mathbf{x})a_{iN_s^-}^-, & \mathbf{x} \in \bar{\Omega}^- \end{cases} \tag{16}$$

where Dirichlet and Neumann type of interface conditions are introduced on the interface in (16) for optimal convergence [11]. For elasticity, (16) refers to as the displacement continuity and traction equilibrium. Localized radial basis functions have been introduced in this subdomain collocation method for enhanced accuracy [11]. An inclusion problem shown in Fig. 7 is solved by radial basis collocation method (RBCM), the subdomain RBCM (SD-RBCM), and subdomain collocation method with localized radial basis functions (SD-LRBCM). Results in Fig. 8 show that RBCM completely missed the materials interface due to its nonlocal approximation, while both SD-RBCM and SD-LRBCM offer much enhanced accuracy.

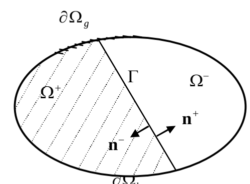

Figure 6: Two subdomains of a problem with heterogeneity

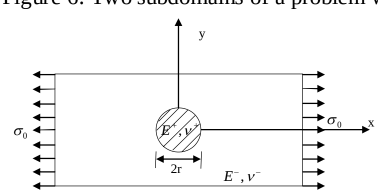

Figure 7: Plate with circular inclusion subjected to horizontal traction

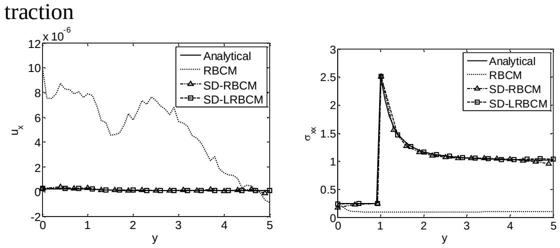

Figure 8: Displacement and stress solutions obtained by RBCM, SD-RBCM, and SD-LRBCM

This subdomain radial basis collocation method has also been applied to fracture mechanics, in which the domain is sub-divided into a near-tip and far-field subdomains as shown in Fig. 9. The optimal dimension for the near-tip subdomain has been derived based on balanced errors at the triple junction [12]. Interface conditions in Eq. (15) are introduced on the interfaces of subdomains. Figure 10 demonstrates a tensile specimen with edge crack analyzed by RBCM, RBCM with visibility criteria

(RBCM-VC), and the proposed SD-RBCM. Again, a superior accuracy in SD-RBCM is shown in Fig. 11.

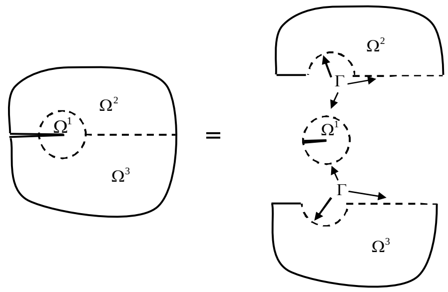

Figure 9: Domain partitioning in crack problem

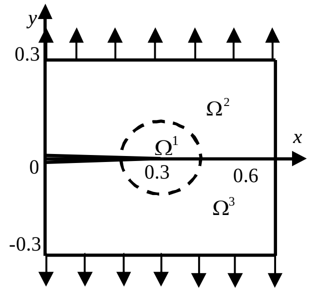

Figure 10: Edge crack in a specimen subjected to tension

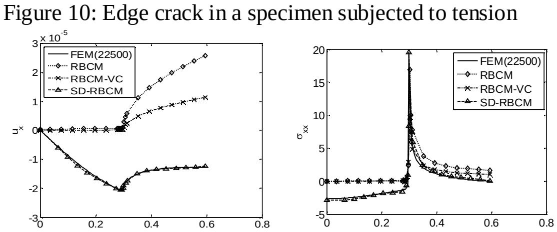

Figure 11: Edge crack plate subjected to tension analyzed by different collocation methods

Next, a reproducing kernel collocation method (RKCM) has been proposed to enhance the stability of RBCM. The error estimate provided in [13] shows that the polynomial degree of greater than one is needed in RKCM for convergence, which is different from the Galerkin type meshfree method, such as RKPM. A stability analysis of RKCM on the conditioning of the discrete system has been derived [14]. Our stability analyses, validated with numerical tests, show that this approach yields a well-conditioned and stable linear system similar to that in the finite element method.

Finally, we use RKPM to model the process of a bullet penetrating through a concrete plate. In this simulation, a micro-crack informed damage model [15] has been used in conjunction with the stabilization methods discussed in Section 2. The numerical and experimental damage patterns of the exit face are compared in Fig. 12, and good agreement is observed.

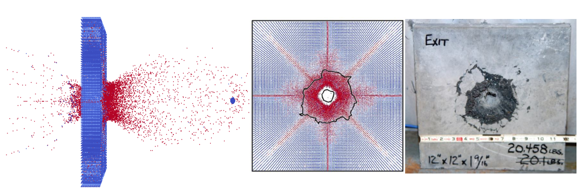

Figure 12: Experimental and numerical damage patterns on the exit face of a concrete plate been penetrated by a bullet

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