A computational model of three-dimensional shape memory alloy trusses

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Abstract

A computational model of three-dimensional trusses made of shape memory alloys is proposed. The model takes into account the characteristic hysteresis loops in the pseudoelastic range of material behaviour. The derived incremental problem takes up the form of a mixed linear complementarity problem (MLCP) and has been solved by a computer program developed by this author. Included are numerical results for the elastic and pseudoelastic Mises truss.

Keywords: smart materials, material properties, trusses, structural mechanics, finite element methods, numerical analysis

1. Introduction

Shape memory alloys (SMAs) are the active (often called intelligent) materials which have found many innovative applications in different fields of science and engineering, including aerospace and mechanical engineering, and bioengineering [7, 2, 5, 1]. This is thanks to the shape memory effect and pseudoelasticity they exhibit, which ordinary metals and alloys do not possess. Pseudoelasticity of a SMA, which is the property being considered in this contribution, means that this material is capable of sustaining large deformations (8-10%) and to retain to its non-deformed shape upon unloading, at temperatures above a certain temperature A_f.

Our aim here is to obtain a workable computational model for three-dimensional trusses made of SMAs in the range of the pseudoelastic material behaviour.

2. Constitutive relations

We will present a computational model for trusses made of shape memory alloys (SMAs) in the range of pseudoelasticity (superelasticity). The idealized pseudoelastic response in a 1D case is shown in Fig. 1, where characteristic hysteresis loops can be observed. This advantageous feature of SMAs is a result of martensitic phase transformations and amounts to large strains of the order of 10%, which are recoverable during mechanical loading-unloading cycles conducted at a constant temperature. Martensitic phase transformation is a first-order reversible transformation from a high temperature phase with greater symmetry (*austenite*) to a low temperature phase with lower symmetry (*martensite*). It can be induced by stresses, changes of temperature, or magnetic fields. We build here on the concepts presented in [6, 8, 4] and consider the simplified case of a three-phase material, i.e. two variants of martensite M_+ and M_-, or $M_m, m = 1, 2$, and austenitic phase A. Let c_1 and c_2 denote a volume fraction of martensite M_1 and M_2, respectively, and $c_3 = 1 - c_1 - c_2$ be the volume fraction of austenite.

The adopted averaged Helmholz free energy, $\widetilde{W}$, of the austenite/martensite mixture is a piecewise parabolic nonconvex function. With the notations: ε – the total strain, η_m – the phase transformation strain of variant m, $\varpi_i(\theta)$ – the free energy at a stress-free state, the free energy can be defined as

$$\widetilde{W}(\varepsilon, \mathbf{c}) = \frac{1}{2} E \left(\varepsilon - \sum_{m=1}^{2} c_m \eta_m \right)^2 + \sum_{i=1}^{3} c_i \varpi_i(\theta) + \sum_{m=1}^{2} \frac{1}{2} B_m (1 - c_m) c_m + I_{[0,1]}(\mathbf{c}) \quad (1)$$

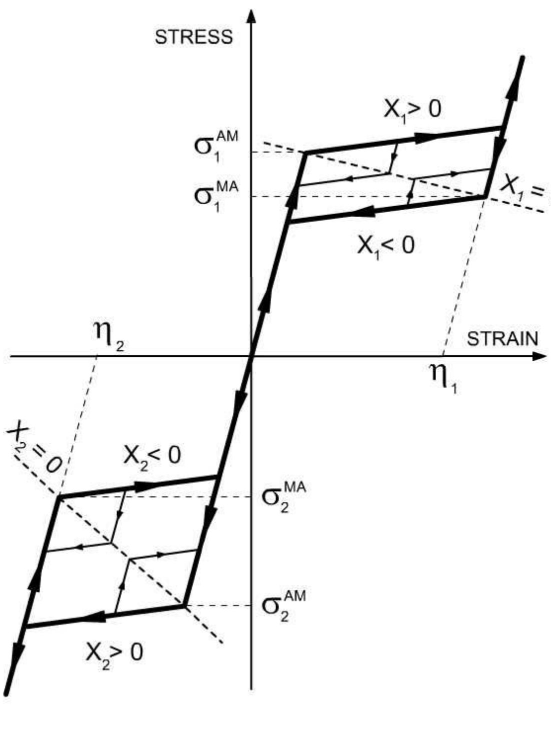

Figure 1: Pseudoelastic hysteretic behaviour

where additionally E, B_m are material parameters, cf [6, 4], and $I_{[0,1]}$ is the indicator function of interval $[0, 1]$.

The axial stress is

$$\sigma \equiv \partial\widetilde{W}/\partial\varepsilon = E(\varepsilon - c_1\eta_1 - c_2\eta_2) \tag{2}$$

The hysteresis loops in Fig. 1 can be described by the following phase transformation rules ($m = 1, 2$)

$$\begin{array}{lll} \text{if} & X_m(\varepsilon, c_m) = \kappa_{3\to m}(c_m) & \text{then } \dot{c}_m \geq 0 \\ \text{if} & X_i(\varepsilon, c_m) = \kappa_{m\to 3}(c_m) & \text{then } \dot{c}_m \leq 0 \\ \text{if} & \kappa_{m\to 3}(c_m) < X_m(\varepsilon, c_m) < \kappa_{3\to m}(c_m) & \text{then } \dot{c}_m = 0 \end{array} \tag{3}$$

where

$$X_m = E(\varepsilon - c_m\eta_m)\eta_m - (\varpi_m - \varpi_3) - \frac{B_m}{2}(1 - 2c_m) - R_m$$

is the driving force for the reversible phase transformation from austenite to two variants of martensite ($A \to M_m$), and $\kappa_{3\to m} \geq 0$, $\kappa_{m\to 3} \leq 0$ are threshold functions. A dot ($\dot{c}$) stands for rates of c, and R_m is an element of subdifferential $\partial I_{[0,1]}$.

3. Incremental problem

In transferring the material point relations (2), (3) to those at the level of a truss bar (element) and the truss as a whole, we use the displacement approach. For a typical element e of the space truss, we have as unknowns its node displacements $\mathbf{q}_e = [u_1\, v_1\, w_1\, u_2\, v_2\, w_2]_e^{\mathrm{T}}$ and can express its nodal forces, $\mathbf{Q}_e = [U_1\, V_1\, W_1\, U_2\, V_2\, W_2]_e^{\mathrm{T}}$, as

$$\mathbf{Q}_e = \mathbf{K}_e\mathbf{q}_e - \mathbf{G}_{1,e}^{\mathrm{T}}c_1 - \mathbf{G}_{2,e}^{\mathrm{T}}c_2 \tag{4}$$

where $\mathbf{K}_e$ is the stiffness matrix and $\mathbf{G}_{m,e}$ depends on the properties of phases M_m and the element's coordinates.

Accounting for the equilibrium conditions and displacement compatibility, which are enforced at the current configuration, as well as the phase transformation criteria (3), we finally formulate the incremental problem, from time level t_{n-1} to time level t_n, for a SMA truss as a mixed (nested) linear complementarity problem (for mLCP see e.g. [3]),

$$\begin{array}{c} \mathbf{D}\,\mathbf{x}_n + \mathbf{y}_n = \mathbf{b}_{n,n-1} \\ \mathbf{x}'_n \geq \mathbf{0}, \quad \mathbf{y}^1_n = \mathbf{0}, \quad \mathbf{y}_n \geq \mathbf{0}, \quad \mathbf{x}_n \cdot \mathbf{y}_n = 0 \end{array} \tag{5}$$

The matrix $\mathbf{D}$, vector of unknowns $\mathbf{x}_n$ and vector of data $\mathbf{b}_{n,n-1}$ in (5) have the following structure

$$\mathbf{D} = \begin{bmatrix} -\mathbf{K} & -\mathbf{G}^{\mathrm{T}} & \mathbf{G}^{\mathrm{T}} & & \\ -\mathbf{G} & -\mathbf{H} & \mathbf{H} & -\mathbf{I} & \\ \mathbf{G} & \mathbf{H} & -\mathbf{H} & & -\mathbf{I} \\ & \mathbf{I} & & & \\ & & \mathbf{I} & & \end{bmatrix}$$

$$\mathbf{x}_n = \left\{\Delta\mathbf{q}_n \;\; \Delta\mathbf{c}^+_n \;\; \Delta\mathbf{c}^-_n \;\; \mathbf{r}^1_n \;\; \mathbf{r}^0_n\right\}^{\mathrm{T}}$$

$$\mathbf{b}_{n,n-1} = \left\{\mathbf{f}_{n,\,n-1} \;\; \mathbf{b}^+_{n-1} \;\; \mathbf{b}^-_{n-1} \;\; \mathbf{1} - \mathbf{c}_{n-1} \;\; \mathbf{c}_{n-1}\right\}^{\mathrm{T}}$$

where $\mathbf{K} = \mathbf{K}_E + \mathbf{K}_G$ is the stiffness matrix of the truss, both the elastic and geometric part, and $\mathbf{H}$ is a global matrix depending on the pseudoelastic properties of SMAs. Further, $\Delta\mathbf{q}_n$ is a vector of finite increments of nodal displacements, $\Delta\mathbf{c}^+_n$, $\Delta\mathbf{c}^-_n$ are vectors of the positive and negative part, respectively, of a finite increment of volume fractions.

The first row of the LCP (5) is the global equilibrium condition, while the next two rows express the phase transformation conditions (3) in global form. The last two rows are due to the constraints imposed on finite increments of vectors $\mathbf{c}^{\pm}_n$, with $\mathbf{r}^1_n$ and $\mathbf{r}^0_n$ playing the role of lagrangian multipliers.

We have developed a computer program in Fortran 95, which solves (5) and updates the configuration by the Newton method.

4. Numerical example

The obtained numerical results for a Mises truss, Fig. 2, are shown in Figs. 3 and 4, where for comparison included is also the elastic response. The data used: $a = 200$ cm, $z_0 = 80$ cm, axial stiffness $EA = 10000$ kN, $B = 0.21$ kN/cm^2, $\eta_1 = 0.07 = -\eta_2$, $\varpi_m - \varpi_3 = 0.595$ kJ/cm^3 ($m = 1, 2$). The logarithmic strain measure was used.

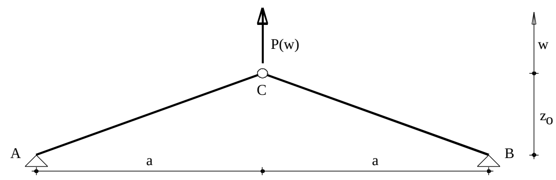

Figure 2: Mises truss

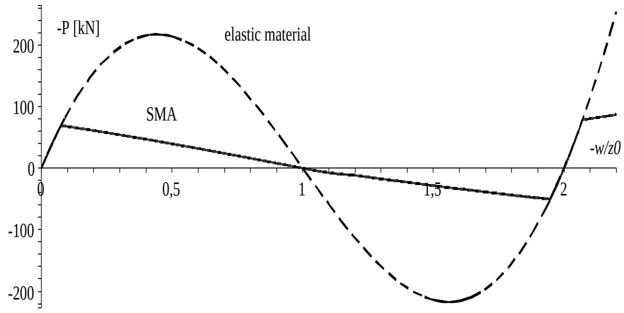

Figure 3: Force P as a function of scaled top node displacement w/z_0 for elastic material (dashed line) and pseudoelastic SMA

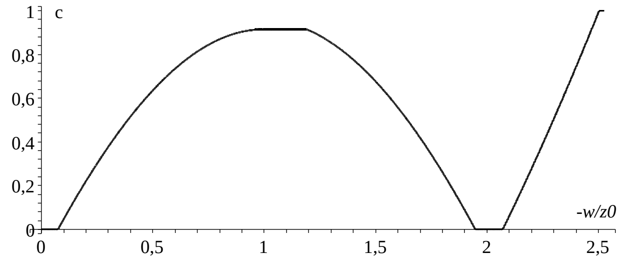

Figure 4: Volume fraction in SMA, $c \in \{c_1, c_2\}$, $c_1 \cdot c_2 = 0$

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