Analysis of piezoelectrically induced ultra-acoustic waves for structural health monitoring

Rolf Lammering[1], **Sven von Ende**[2], **and Bianca Hennings**[1 *]

[1]*Institute of Mechanics, Helmut-Schmidt-University / University of the Federal Armed Forces Hamburg*
Holstenhofweg 85, 22043 Hamburg, Germany
e-mail: rolf.lammering@hsu-hh.de, bianca.hennings@hsu-hh.de

[2]*Rolls-Royce Deutschland Ltd & Co KG*
Eschenweg 11, Dahlewitz, 15827 Blankenfelde-Mahlow, Germany
e-mail: sven.vonende@rolls-royce.com

Abstract

In order to be able to properly analyze piezoelectrically induced high frequency waves in thin plate and shell structures for structural health monitoring purposes, a semi-analytical method as well as an adaptation of the well known finite element method, the spectral finite element method, are presented. The semi-analytical method uses the Fourier transform to solve the Lamé-Navier differential equations in the frequency domain. The inverse transform gives the displacement field in the time domain. Starting point of the spectral finite element method is the semi-discrete equation of motion. High-order finite elements with a special integration scheme are used. Due to a diagonal mass matrix the special requirements of computational wave propagation analysis such as high discretization with respect to space and time are met. The results of an exemplary analysis of both methods are compared.

Keywords: waves, structural monitoring, smart materials, finite element method, structural mechanics

1. Introduction

In smart structures research special focus is currently put on structural health monitoring (SHM) especially with regard to further development of lightweight structures. Lamb waves, i.e. guided elastic waves, are commonly considered as an appropriate means for damage detection in thin plate and shell structures since their propagation is disturbed at damage locations. Especially when Lamb waves are generated at high frequencies, reflections, refractions, and mode conversions occur and are distinct indications for structural damage, see e.g. [3, 4, 5]. For the generation of Lamb waves piezoelectric actuator systems are easy to integrate into the structure and piezoelectric material is likewise used for the sensor system. Beside the ease of structural integration, even very high frequency elastic waves can be generated by the actuators and captured by the sensors.

2. Semi-analytical solution

2.1. *Problem description*

A piezoelectric actuator is bonded to the top surface of an infinite isotropic elastic plate, see Fig. 1 and used for Lamb wave generation. The displacements of the plate are described by the Lamé-Navier differential equations, which are derived under the assumption of small strains from Newton's second law, in which Hooke's law is introduced for stress elimination. Assuming a plane strain state as well as a time harmonic solution and using the conclusion of Helmholtz's theorem, that a vector field can be separated in a curl-free scalar field φ and a divergence-free vector field $\mathbf{\Psi}_i$, the governing equations are written as

$$\partial_{xx}\varphi + \partial_{yy}\varphi + \omega^2 \frac{\rho}{2\mu - \lambda}\varphi = 0 \tag{1}$$

$$\partial_{xx}\mathbf{\Psi}_z + \partial_{yy}\mathbf{\Psi}_z + \omega^2 \frac{\rho}{\mu}\mathbf{\Psi}_z = 0 \tag{2}$$

where λ and μ are the Lamé constants, ρ is the density, ω the angular frequency and $\partial_{(..)}$ the partial derivative with respect to the indicated coordinates.

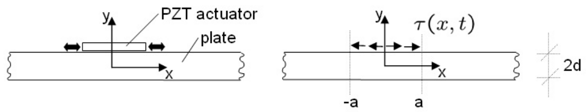

Figure 1: Piezoelectric actuator adhered to plate surface

For the solution of this set of partial differential equations the boundary conditions are essential. In Fig. 2 the assumed stress distribution between the piezoelectric actuator and the plate is shown. The shear stress is concentrated in the small range $[a, b]$ at the edge of the actuator. If this range becomes small, $a\tau_0$ can be interpreted as a pin force. Furthermore, the out-of-plane stress is equal to zero at the surface of the plate. This leads to the following boundary conditions:

$$\begin{aligned} 0 =& (\lambda + 2\mu)\partial_{yy}\{\varphi[x, y = d]\} - 2\mu\partial_{xy}\{\mathbf{\Psi}_z[x, y = d]\} \\ &+ \lambda\partial_{xx}\{\varphi[x, y = d]\} \end{aligned} \tag{3}$$

$$\begin{aligned} \tau[x] =& \mu(\partial_{yy}\{\mathbf{\Psi}_z[x, y = d]\} + 2\partial_{xy}\{\varphi[x, y = d]\} \\ &- \partial_{xx}\{\mathbf{\Psi}_z[x, y = d]\}) \end{aligned} \tag{4}$$

The set of equations (1) to (4) hence describes the complete mechanical problem.

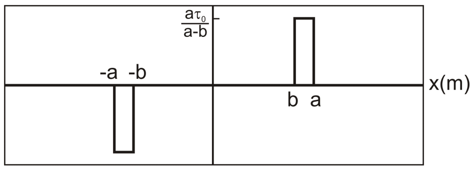

Figure 2: Shear stresses between actuator and plate surface

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2.2. *Solution for transient excitation*

In the case of harmonic excitation, the shear stresses in the interface between actuator and plate are described by

$$\tau[x,t] = \tau[x]e^{i\omega_0 t}\,. \tag{5}$$

For the solution of Eqns. (1) to (4), a Fourier transform is applied. The resulting ordinary differential equations are solved in the wavenumber domain. Symmetric and antisymmetric solutions are found with respect to the mid-surface of the plate. The solution in the time domain is obtained by the inverse Fourier transform through application of the residue theorem, cf. [2].

In the case of transient loading, the shear stresses are written in a more general form, namely

$$\tau[x,t] = \tau[x]f[t]\,. \tag{6}$$

Again, a Fourier transform is applied in order to find a solution in the frequency domain. The inverse Fourier transform gives the displacement field in the time domain. Two difficulties have to be overcome. First, a continuous solution $u_i[x,y,\omega]$ has to be found. However, the roots of the related Rayleigh-Lamb frequency equation cannot be evaluated analytically and thus the derivation of a continuous solution is difficult. Therefore, the roots are found by use of numerical algorithms for discrete values ω_n. Secondly, direct integration is not possible in most cases. Therefore numerical integration algorithms are used in order to obtain the displacement field in the time domain, cf. [2].

3. Finite element approximation

3.1. *Problem description*

In the numerical analysis of wave propagation problems, the well established finite element method is also a powerful tool. However, a high spatial resolution of the finite element mesh has to be assured in order to be able to account for small wavelengths. Moreover, extremely small time steps have to be chosen in order to allow for the analysis of wave propagation hat high frequencies, e.g. at $100kHz$ and above. Like in structural dynamics, starting point is the semi-discrete differential equation of motion

$$\mathbf{Ku} + \mathbf{M\ddot{u}} = \mathbf{F}[t] \tag{7}$$

which has to be solved with respect to the displacement field $\mathbf{u}$. Here, $\mathbf{K}$ and $\mathbf{M}$ denote the stiffness and mass matrices, respectively, and $\mathbf{F}[t]$ stands for the time dependent forces.

3.2. *Spectral finite element solution*

The spectral finite elements method is a promising approach to adapt the conventional finite element method to the above-mentioned special situation of Lamb wave analysis. In comparison to well-known finite elements spectral finite elements apply high-order shape functions. The basis of these shape functions are Lagrangian polynomials L_j^n which read in the one-dimensional case with coordinate ξ

$$L_j^n(\xi) = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\xi-\xi_k}{\xi_j-\xi_k}. \tag{8}$$

Here, n is the degree of the polynomial and j is the node with non-vanishing function value. The most important characteristic of this kind of shape functions is the arbitrary position of the interpolation points. In the case of spectral finite elements, the interpolation points are located at the Gauss -Lobatto-Legendre (GLL) points which are obtained by the solution of $(1-\xi^2)P'_{n-1}(\xi) = 0$. Here, P'_{n-1} denotes the derivative of the Legendre polynomial of degree $n-1$. The corresponding integration scheme is based on the Gaussian integration rule with two fixed integration points at the interval boundaries [-1, 1], cf. Fig. 3.

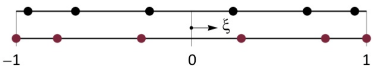

Figure 3: Exemplary location of Gauss (top) and Gauss-Lobatto-Legendre points (bottom).

The GLL quadrature results in a diagonalized mass matrix and thus offers an advantage with respect to the computational costs. At time step $t+\Delta t$, the resulting equation of motion reads $\mathbf{K_{eff}}\mathbf{u_{t+\Delta t}} = \mathbf{F_{eff}}$. It can efficiently be solved since $\mathbf{K_{eff}}$ is also obtained as a diagonal matrix:

$$\mathbf{K_{eff}} = \frac{\mathbf{M}}{\beta\Delta t^2} + \frac{\mathbf{C}}{2\Delta t}. \tag{9}$$

4. Analysis of ultra-acoustic waves

The performance of spectral finite elements is tested with a steel plate of $4.8mm$ thickness. A cut through the plate is investigated in the plane strain state and an exemplary computation with 480 degrees of freedom (20 elements with 9x3 nodes) is carried out. Even at this low number of nodes across the thickness, a good agreement with the semi-analytical solution is achieved, cf. Fig. 4. Further computations which include higher order elements as well as structural defects are not shown here for brevities sake.

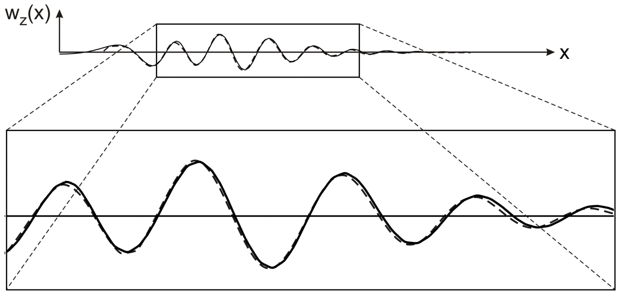

Figure 4: Comparison of semi-analytical (dashed line) and numerical solution (continuous line).

5. Conclusion and outlook

A semi-analytical as well as a numerical method for the analysis of high-frequency elastic waves have been investigated. Future work will focus on carbon fiber reinforced polymers as well as on further enhancement of the computational efficiency.

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