Simple displacement norm matrices for numerical evaluation the inf-sup condition for beams, plates and shells

Wojciech Gilewski and Marta Sitek

Department of Structural Mechanics and Computer Aided Engineering, Warsaw University of Technology
al. Armii Ludowej 16, 00-637 Warsaw, Poland
e-mail: w.gilewski@il.pw.edu.pl
e-mail: m.sitek@il.pw.edu.pl

Abstract

Development of high-performance finite elements for thick, moderately thick as well as thin shells and plates is one of the active areas of the finite element technology for 40 years, followed by hundreds of publications. A variety of shell elements exist in FE codes, but “the best” finite element is still to be discovered. The work deals with an evaluation of some existing beam, plate and shell finite elements, from the point of view of the third of three requirements to be satisfied by a finite element: ellipticity, consistency and inf-sup condition. It is difficult to prove the inf-sup condition analytically, so a numerical verification is analysed. Two norm matrices are considered. It is shown that the norm matrix equivalent to the mass matrix with unit mass density can be used for evaluation of the test. Finite elements from various computer systems can be evaluated with the use of the proposed simple norm matrices.

Keywords: finite element methods, shells, plates, beams

1. Introduction

Shells and plates are widely considered in engineering applications. The corresponding discretization procedures are not yet sufficiently reliable, in particular as regards to shell structures [3, 6]. In a formulation we should aim to satisfy [2]:
Ellipticity. This condition ensures that the finite element model is solvable and physically means there are no spurious zero energy modes [2, 5].
Consistency. The finite element solution must converge to the solution of a mathematical problem when element size h is close to zero [2, 5].
Inf-sup condition. Satisfying this condition implies uniform and optimal convergence in bending-dominated shell problems. In general it is very difficult to prove analytically whether a shell or plate finite element satisfies this condition and numerical tests are to be employed [1, 4, 5].

A variety of shell finite elements exist in commercial FE codes and, according to the authors’ opinion, it is the right time to compare and evaluate them. The tests presented below can be used for this evaluation and comparison.

The present paper is dedicated to the numerical verification of the inf-sup condition with the use of two displacement norm matrices. One of them is equivalent to the mass matrix with the unit density and can be easily used for evaluation of finite elements from commercial FE codes.

2. Numerical evaluation of the inf-sup condition

Shell elements in the finite element systems are based on the displacement, mixed, assumed strain or other formulation. Details of the inf-sup conditions and the concept of the inf-sup test for all models can be found in [1, 4, 5]. Effective implementation of the inf-sup test consists of the following steps:

- A sequence of N finite element meshes should be chosen for a selection of bending dominated problems, with decreasing characteristic element size h^k ($k=1,2,\ldots,N$). It is preferable to use the element sides not aligned with the asymptotic lines of the mid-surface.
- For every k in the sequence, establish the stiffness matrix $\hat{\mathbf{K}}^k$ and the norm matrix $\mathbf{S}^k$ and calculate the smallest eigenvalue $\lambda^k_{\min}$ of the generalized eigenvalue problem
  $\hat{\mathbf{K}}^k\mathbf{q} = \lambda^k\mathbf{S}^k\mathbf{q}$
- Plot $\log\left(\lambda^k_{\min}\right)$ versus $\log\left(h^k\right)$.
- If the curve clearly flattens out as h^k decreases (the $\lambda_{\min}$ value stabilizes at some positive level) then the inf-sup condition is satisfied.
- If the right behaviour is observed for all test problems, the element passes the inf-sup test.

3. Selection of norm matrices

The norm matrix $\mathbf{S}$ can be selected from one of the following:

$$\mathbf{S}_0 = \int_\Omega \mathbf{N}_0^T\mathbf{n}_0\mathbf{N}_0 d\Omega, \tag{1}$$

$$\mathbf{S}_{01} = \mathbf{S}_0 + \int_\Omega \mathbf{N}_1^T\mathbf{n}_1\mathbf{N}_1 d\Omega, \tag{2}$$

where $\mathbf{N}_0$ are standard displacement shape functions, $\mathbf{N}_1$ are extended shape functions with respective derivatives, $\mathbf{n}_0$, $\mathbf{n}_1$ are diagonal matrices equivalent to the unit translational and rotational matrix distribution, Ω is the area of integration (depending on the structure – beam, plate or shell). For example, for a Timoshenko beam finite element we have

$$\mathbf{n}_0 = \begin{bmatrix} A & 0 \\ 0 & J \end{bmatrix},\ \mathbf{n}_1 = \begin{bmatrix} A & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & A \end{bmatrix}. \tag{3}$$

The norm matrix $\mathbf{S}_0$ is equivalent to the mass matrix of the finite element with unit mass density: $\mathbf{S}_0 = \mathbf{M}$ with $\rho = 1$.

The following beam examples (Fig.1 and 2) show that the convergence of the inf-sup condition with the use of the $\mathbf{S}_0 = \mathbf{M}$ norm matrices is similar to the convergence obtained with the use of the more complicated $\mathbf{S}_{01}$ matrices. Similar results were received for selected plate finite elements.

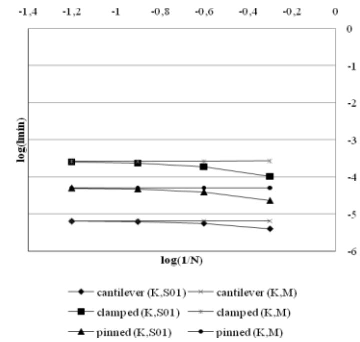

Figure 1: Inf-sup condition thin beam with the use of finite elements with physical shape functions [5]. Two norm matrices used. Three boundary conditions applied

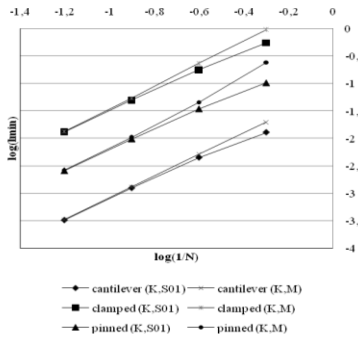

Figure 2: Inf-sup condition thin beam with the use of finite elements with linear shape functions [5]. Two norm matrices used. Three boundary conditions applied

4. Examples for shell finite elements

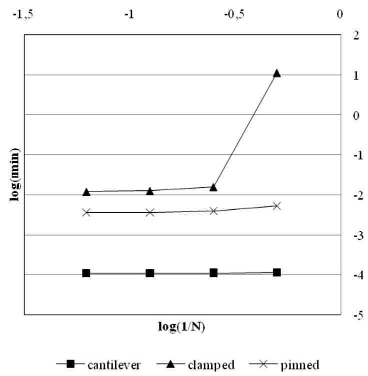

Figure 3: Inf-sup condition for ABAQUS finite element S4R5 - plate test cases

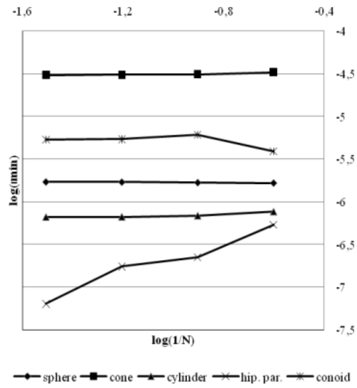

Figure 4: Inf-sup condition for ABAQUS finite element S4R5 - shell test cases

Two numerical examples of evaluation of the inf-sup test with the use of $\mathbf{S}_0 = \mathbf{M}$ matrices are presented in Fig. 3, 4. The ABAQUS finite element named S4R5 is used for plates with three boundary conditions (clamped, simply supported and cantilever) and five shell geometries: sphere, cone, cylinder, paraboloid and conoid [5]. More examples will be presented during the conference.

5. Concluding remarks

The present paper is dedicated to proof that the simple displacement norm matrix $\mathbf{S}_0 = \mathbf{M}$ can be used for numerical evaluation of the inf-sup test for finite elements. The use of this matrix allows the analysis of correctness of various finite elements that exist in FE codes without additional programming.

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