Cylindrical Element based on strain approach

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Abstract

This paper studies the formulation of cylindrical element based on strain approach. This derivation illustrates the basic steps in the application of the strain model to the construction of high performance sufficient, nonconforming elements with drilling rotation. Numerical experiments have been conducted to assess the accuracy and reliability of the cylindrical element, compared to the theoretical results, and other cylindrical finite element.

Keywords: numerical analysis, elasticity, finite element method, shells, structural mechanics.

1. Introduction

The strain functions and displacement functions of [1] are the starting point. In this paper a strain based cylindrical element with additional rotation SBCEAR is developed. This element is again rectangular in plan with corner nodes only and six degrees of freedom at each node. These degrees of freedom are taken to be the essential degrees of freedom used in strain element of [1] plus an additional degree of freedom of rotation.

1.1 The strain-displacement equations used in [1]:

The displacement u, v and w are shown on figure 1. The strain-displacement equations (1) are taken from [7]

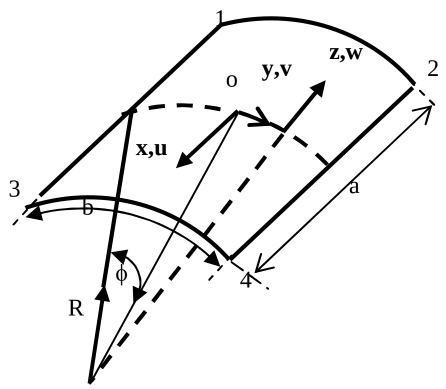


Figure 1: An element with coordinate system. ϕ=y/R

The strain – displacement equations are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad (1a)$$

$$\varepsilon_y = \frac{\partial v}{R\partial \phi} + \frac{w}{r} \qquad (1b)$$

$$\varepsilon_{xy} = \frac{\partial u}{r\partial \phi} + \frac{\partial v}{\partial x} \qquad (1c)$$

$$K_x = -\frac{\partial 2w}{\partial x^2} \qquad (1d)$$

$$K_y = -\frac{\partial 2w}{r^2\partial \phi^2} + \frac{\partial v}{r^2\partial \phi} \qquad (1e)$$

$$K_{xy} = -\frac{\partial 2w}{r\partial x\partial \phi} + \frac{\partial v}{r\partial x} \qquad (1f)$$

By eliminating u, v, w from the above equations we obtain the compatibility equations (2):

$$\frac{\partial^2 \varepsilon_\phi}{\partial x^2} + \frac{k_x}{r} + \frac{\partial^2 \varepsilon_x}{r^2\partial^2\phi} - \frac{\partial^2 \varepsilon_{x\phi}}{r\partial x\partial \phi} = 0 \qquad (2a)$$

$$\frac{\partial^2 \varepsilon_\phi}{\partial x^2} + \frac{k_x}{r} + \frac{\partial^2 \varepsilon_x}{r^2\partial^2\phi} - \frac{\partial^2 \varepsilon_{x\phi}}{r\partial x\partial \phi} = 0 \qquad (2b)$$

$$\frac{\partial k_x}{\partial x} - \frac{\partial k_{x\phi}}{r\partial \phi} = 0 \qquad (2c)$$

These equations can be satisfied by putting:

$$\frac{\partial \varepsilon_x}{r\partial \phi} = \frac{\partial \varepsilon_{x\phi}}{\partial x} \qquad (3a)$$

$$\frac{\partial^2 \varepsilon_\phi}{\partial x^2} = -\frac{k_x}{r} \qquad (3b)$$

$$\frac{\partial k_x}{r\partial \phi} = \frac{\partial k_{x\phi}}{\partial x} \qquad (3c)$$

$$\frac{\partial k_\phi}{\partial x} = \frac{\partial k_{x\phi}}{r\partial \phi} \qquad (3d)$$

Firstly we integrate equations (1) with all strains equal to zero to obtain the component of rigid body displacement u_r, v_r and w_r (4).

$$u_r = r\,\alpha_2 \cos\phi + r\alpha_4 \sin\phi + \alpha_5 \qquad (4a)$$

$$v_r = (\alpha_1+\alpha_2)\sin\phi - (\alpha_1+\alpha_2)\cos\phi + \alpha_6 \qquad (4b)$$

$$w_r = -(\alpha_1+\alpha_2 x)\cos\phi - (\alpha_2+\alpha_4 x)\sin\phi \qquad (4d)$$

Secondly the remaining 14 constants are available for expressing the strains as:

$$\varepsilon_x = \alpha_7 + \alpha_8\phi \quad (5a)$$

$$\varepsilon_y = \alpha_9 + \alpha_{10}x + [-\alpha_{12}x^2/2r - \alpha_{13}x^3/6r - \alpha_{14}x^2\phi/2r - \alpha_{16}x^3\phi/6r] \quad (5b)$$

$$\varepsilon_{xy} = \alpha_{11} + [\phi_8 x/r] \quad (5c)$$

$$K_x = \alpha_{12} + \alpha_{13}x + \alpha_{14}\phi + \alpha_{15}x\phi \quad (5d)$$

$$K_y = \alpha_{16} + \alpha_{117}x + \alpha_{18}\phi + \alpha_{19}x\phi \quad (5e)$$

$$K_{xy} = \alpha_{20} + [\alpha_{14}x/R + \alpha_{15}x^2/2r + \alpha_{17}r\phi + \alpha_{19}r\phi^2/2] \quad (5f)$$

1.2 Development of the present element "SBCEAR"

In this section a strain based cylindrical element is developed. This element is again rectangular in plan with corner nodes only and six degrees of freedom at each node. These degrees are taken to be the essential degrees of freedom used in [1] plus an additional degree of freedom of rotation . If the new element is to have 24 degrees of freedom then the assumed strains should be expressed in terms of 18 independent constants. These constants are apportioned as below:

$$\varepsilon_x = \alpha_7 + \alpha_8\phi + \alpha_{21}\phi^2 + 2\alpha_{22}x\phi^3 + [2r\alpha_{23}\phi] \quad (6a)$$

$$\varepsilon_y = \alpha_9 + \alpha_{10}x + [-\alpha_{12}x^2/2r - \alpha_{13}x^3/6r - \alpha_{14}x^2\phi/2r - \alpha_{16}x^3\phi/6r] \; {}_{-}\alpha_{21}x^2 - 2\alpha_{22}x^3\phi \quad (6b)$$

$$\varepsilon_{x\phi} = \alpha_{11} + [\phi_8 x/r] + 2\alpha_{23}x + 2\alpha_{24}\phi + [2\alpha_{21}x\phi + 3\alpha_{22}x^2\phi^2]/r \quad (6c)$$

$$K_x = \alpha_{12} + \alpha_{13}x + \alpha_{14}\phi + \alpha_{15}x\phi + r[2\alpha_{21} + 12\alpha_{22}x\phi] \quad (6d)$$

$$K_\phi = \alpha_{16} + \alpha_{117}x + \alpha_{18}\phi + \alpha_{19}x\phi \quad (6e)$$

$$K_{x\phi} = \alpha_{20} + \alpha_{14}x/r + \alpha_{15}x^2/2r + \alpha_{17}r\phi + \alpha_{19}r\phi^2/2 + [6\alpha_{22}x^2] \quad (6f)$$

The bracketed terms are then added so that equations (2) or (3) are satisfied.
Expressions (6) are equated to the corresponding expressions in terms of u, v and w from equations (1) which give after integration the component of displacement due to straining u_s, v_s and w_s, this added to the first component to obtain the final shape function u, v and w hence.

$$u = r\alpha_2\cos\phi + r\alpha_4\sin\phi + \alpha_5 + \alpha_7(x) + \alpha_8(x\phi) + \alpha_{11}(r\phi) + \alpha_{17}(r^3\phi^2/2) + \alpha_{19}((r^3\phi - r^3\phi^3)/6) + \alpha_{20}(-r^2\phi) \quad (7a)$$

$$v = (\alpha_1+\alpha_2)\sin\phi - (\alpha_1+\alpha_2)\cos\phi + \alpha_6 + \alpha_{16}(r^2\phi) + \alpha_{17}(r^2\phi x) + \alpha_{18}(r^2\phi^2/2) + \alpha_{19}(-r^2x + r^2x\phi^2/2) + \alpha_{20}(rx) \quad (7b)$$

$$w = -(\alpha_1+\alpha_2x)\cos\phi - (\alpha_2+\alpha_4x)\sin\phi + \alpha_9(r) + \alpha_{10}(rx) + \alpha_{12}(-x^2/2) + \alpha_{14}(-\phi x^2/2) + \alpha_{15}((-\phi x^3/6) + \alpha_{16}(-r^2) + \alpha_{17}(-r^2x) + \alpha_{18}(-r^2\phi) + \alpha_{19}(-r^2\phi x) + \alpha_{20}(-x^3/2) \quad (7c)$$

$$\phi_y = \alpha_2(\cos\phi) + \alpha_4(\sin\phi) + \alpha_{10}(r) + \alpha_{12}(-x) + \alpha_{14}(-\phi x) + \alpha_{15}(-\phi x^2/2) + \alpha_{17}(-r^2) + \alpha_{19}(-r^2\phi) + \alpha_{20}(-3x^2/2) \quad (7d)$$

$$\phi_x = (w - v)/r = \alpha_6(-1/r) + \alpha_{14}(-x^2/2r) + \alpha_{15}((-x^3/6r) + \alpha_{16}(-r\phi) + \alpha_{17}(-rx\phi) + \alpha_{18}(-r\phi^2/2 - r) + \alpha_{19}(-rx\phi^2/2) + \alpha_{20}(-x) + \alpha_{22}(-2x^3) \quad (7e)$$

$$\phi_z = (\partial v/\partial x - \partial u/\partial y)/2 = \alpha_2(\sin\phi) + \alpha_4(-\cos\phi) + \alpha_8(-x/2r) + \alpha_{11}(-1/2) + \alpha_{17}(r^2\phi) + \alpha_{19}(r^2\phi^2/2 - r^2) + \alpha_{20}(r) + \alpha_{21}(-x\phi/r) + \alpha_{22}(-3x^2\phi^2/2r) + \alpha_{23}(-x) + \alpha_{24}(-\phi) \quad [7f]$$

2. Numerical examples

Performance of the cylindrical shell finite elements developed is evaluated by working through four numerical examples, pinched cylinder problem with free edges, pinched cylinder problem with diaphragm, curved beam with Static Loads and cylindrical panel subjected to its own weight. A summary of results is provided in Figures 3-9. Again, the SBCE element generates displacements that are closer to the theoretical solution as predicted by the best shell elements [1]. Selected convergence curves for thick pinched cylinder problem with free edges (h=0.094 in), show that the present element and that of [1] provide much greater accuracy and is in good agreement with the theoretical solution, shown in Figure 4.

*2.1. **Pinched cylinder problems** with free and diaphragm edges*

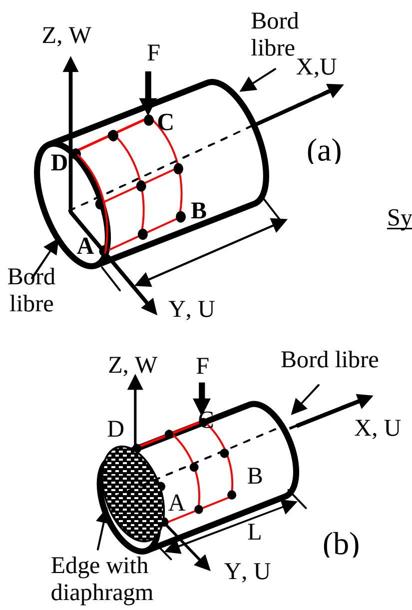
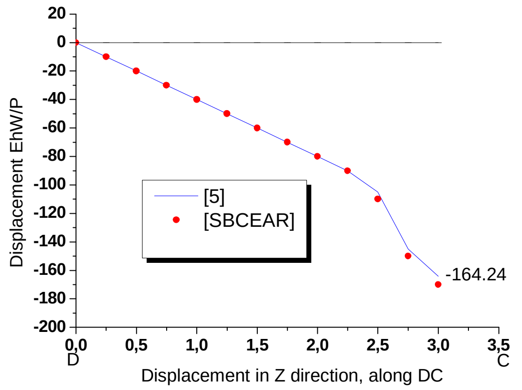


Figure 2 a,b : The pinched cylinder, with one octant shown with free and diaphragm edges



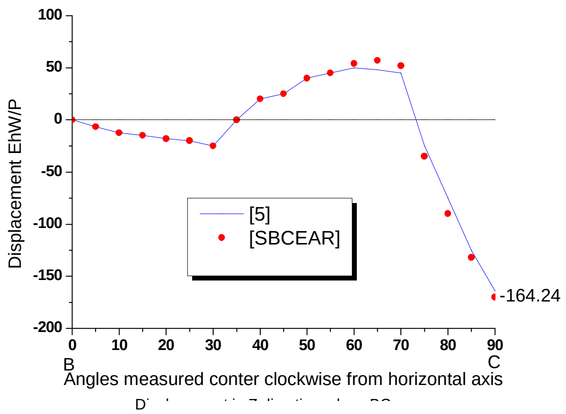


Figure 3: The normal displacements Wc along DC and BC for pinched cylinder problems with rigid diaphragm

Also we can see that the displacement computed by SBCE ARelement and that of [5] are in very close agreement.

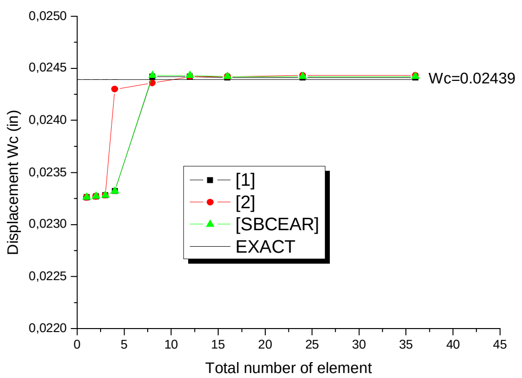


Figure 4: Convergence of displacement Wc for thin pinched cylinder with freeedges (h=0.01548 in)

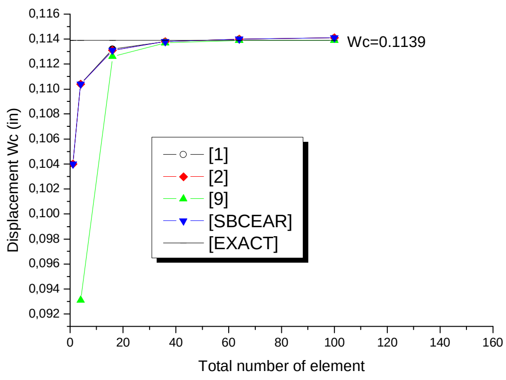


Figure5: Convergence of displacement Wc for thick pinched cylinder with free edges (h=0.094 in)

2.2. Shell- curved beam with static loads

In this example a curved cantilever beam, modeled with shell element, is subjected to unit forces at the tip in the in-plane and out-of-plane directions, that is, the Y, Z directions, respectively. The in-plane and out-of-plane loads are applied in different load cases. The tip displacements in the direction of the load are compared with analytical solution as shown in figure 6.

<u>Data:</u>
Rin=4.12
Rex=4.32
R=(Rin+Rex)/2
t=0.2
E=10+7
v=0.25
Arc=90°

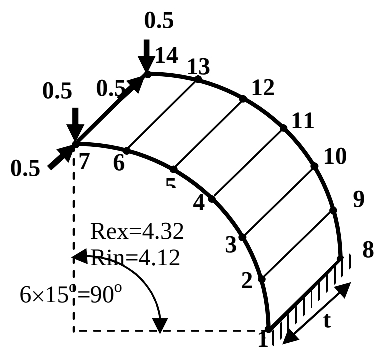
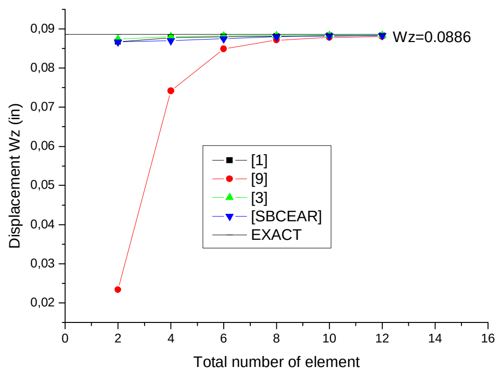


Figure 6: Shell-Curved Beam with Static



The numerical results for the " SBCEAR " element are

Figure 7: Convergence of displacement Wz for Shell-Curved Beam with Static Loads.

compared against the theoretical solution and numerical results reported in the literature for the performance of other elements, shown in figure 7. The results from this test indicate that, with the same meshes, the " SBCEAR " element gives more accurate results than [3] and rapidity convergent results as the best element [1], shown in figure 7.

3.3. *Cylindrical panel subjected to its own weight*

Other authors have used a barrel vault problem as a test for finite elements. The straight edges are free and the curved edges are supported by diaphragms. The geometry, properties and loading are indicated on figure 8: (R/h = 100, L/h = 200). The transverse shearing strains are negligible where the membrane strains are important compared to those of bending. This problem is used as aptitude test of an element to simulate membrane state of stresses (strains). The reference solution based on the shallow shell theory [8] and that based on the deep shells theory [9] is appreciably different. The quarter of the roof is discredited by considering regular grids with N=4, 6 and 8 elements on edges AB and AD.

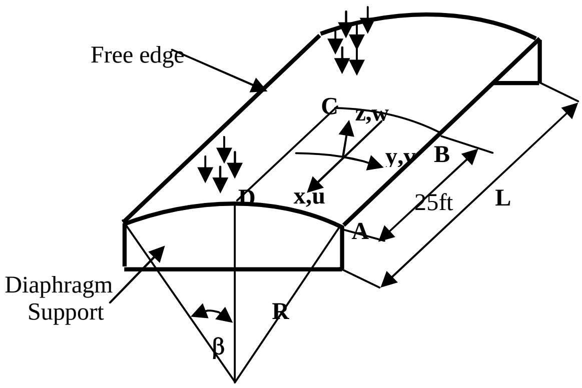


Figure 8: The barrel vault problem, with a uniform gravity loding 90 lb/ft^{-2} of shell area

Data:
L= 50ft; R = 25 ft; h = 0, 25 ft; $\beta = 40°$
$E=432 \times 10^6\ Pa/ft^2$; $\nu = 0$; $f_z = -90 lb/ft^2$
Boundary conditions
$v=w== \theta_x = 0$ at AD
Symmetry conditions
$v=\theta_X=0$ at CD
$u= \theta_Y=0$ at CB
Reference values (Deep shell theory):
$w_B=3.70$ in
Analytical values (Shallow shell theory):
$w_B=3.58$ in

.Conclusion

Using the strain approach for the development of the new element SBCE leads to the exact representation of rigid body motions, also leads to higher order polynomial terms without the need for the addition of internal degrees of freedom and has allowed avoiding using finite element of higher order.
Numerical results obtained, using this element, agree well with those from other investigation and theoretical results.

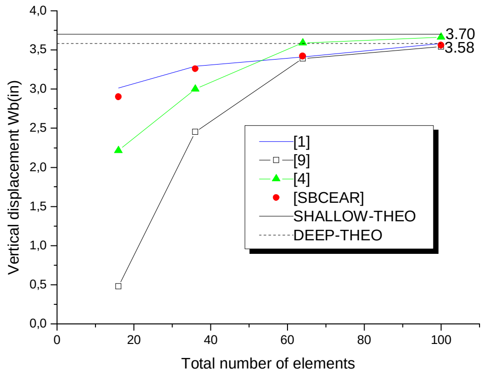


Figure 9: Convergence of vertical displacement W_B, For barrel vault using total number of elements.

5. References

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5. APPENDICES:

In calculating the element stiffness matrix we require the integral

$$\left[K_0\right] = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} Q^T D Q\,dx\,dy$$

Where the strain matrix Q of [1] is:

$$[Q] = \left[\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & & x/r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2/2r & x^3/6r & x^2\phi/2r & x^3\phi/6r & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & \phi & x\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x & \phi & x\phi & 0 \\ 0 & 0 & 0 & x/r & x^2/2r & 0 & r\phi & 0 & r\phi^2/2 & 1 \end{matrix}\right]$$