Application of dynamic relaxation in material point method

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Abstract

One of possible ways of solving quasi-static equilibrium problems is the dynamic relaxation method. Instead of an original problem which can be hard to solve, e.g. because of troubles with convergence in iterative procedure, the dynamic problem with artificial dumping terms can be solved. When these terms are properly chosen, the dynamic approach can be much more computationally efficient than the quasi-static one. The application of the dynamic relaxation approach in the material point method, the arbitrary Lagrangian–Eulerian formulation of the finite element method, is considered in the paper. The usefulness of the method is demonstrated in the case of elastic problems with small and large deformations.

Keywords: meshless methods, finite element methods, dynamics, elasticity

1. Introduction

The use of the dynamic relaxation procedure in the material point method (MPM) is considered. The material point method, which is a variant of the finite element method formulated in an arbitrary Lagrangian–Eulerian description of motion, is usually used in analyses of dynamic problems that can be relatively easily solved by the use of an explicit time integration algorithm which does not require assembling of the stiffness matrix. The application of MPM to quasi-static problems seems to be more complicated as constructing the stiffness matrix, when an Eulerian computational mesh is used and the state variables are defined for material points defined independently of the mesh, is somewhat cumbersome, [3, 2]. The dynamic relaxation method (e.g. [6]) allows to solve a quasi-static problem by adding to the (static) equations inertia and dumping terms and solving a corresponding dynamic problem. The dynamic analysis should be terminated when velocities become zero or sufficiently small in order to get an equivalence of the two formulations: the quasi-static and dynamic ones, at least with respect to an assumed tolerance.

Application of the dynamic relaxation technique to MPM has been proposed in [4]. In [1], quasi-static solutions have been found using MPM by applying loads as slowly increasing functions in time.

2. Material point equations and relaxation approach

In the case of a dynamic analysis, the variational formulation can be set in the form of the following equation:

$$\int_{\Omega} \varrho(a_i v_i + \frac{1}{\varrho}\sigma_{ij} v_{i,j})\mathrm{d}x = \int_{\Omega} \varrho\, b_i v_i \mathrm{d}x + \int_{\Gamma_\sigma} t_i v_i \mathrm{d}s \;\; \forall \boldsymbol{v} \in V_0 \quad (1)$$

where V_0 denotes the space of kinematically admissible fields of displacements. The right hand side terms in the above equation are related to the volumetric and surface forces. In MPM, the motion of material points defined independently of the computational mesh is traced by means of interpolation functions defined on the mesh. After use of the standard finite element interpolation procedure, the equation of virtual work (1) leads to the following system of equations for vector of nodal accelerations $\mathbf{a}$:

$$\mathbf{M}\,\mathbf{a} = \mathbf{F} - \mathbf{R}, \quad (2)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{F}$ and $\mathbf{R}$ are the vectors of external and internal nodal forces, respectively. The details can be found in [5]. The system of equations (2) can be solved incrementally by means of an explicit Euler algorithm.

The quasi-static problem which is to be solved can be set in the variational form setting $a_i = 0$ in Eq. (1), and its discrete variant can be obtained setting $\mathbf{a} = \mathbf{0}$ in the left hand side of Eq. (2). When the dynamic relaxation method is used, a dumping term, $\mathbf{C}\,\mathbf{v}$, is introduced into Eq. (2), so, the equation becomes

$$\mathbf{M}\,\mathbf{a} + \mathbf{C}\,\mathbf{v} = \mathbf{F} - \mathbf{R} \quad (3)$$

where $\mathbf{C}$ is the matrix of dumping coefficients and $\mathbf{v}$ the vector of nodal velocities. The choice of setting the values of the dumping coefficients is determined by computational efficiency. When the system (3) is linear and $\mathbf{F} = \mathbf{0}$, its modal decomposition (with assumption that $\mathbf{C} = \alpha\,\mathbf{M}$) gives the following set of equations: $\bar{a}_i + \alpha_i \bar{v}_i + \omega_i^2 \bar{u}_i = 0$ for the ith eigenvector $\bar{u}_i$, and the critical dumping factor can be calculated from the formula: $\alpha_i = 2\omega_i$. In the present paper, the dumping terms have been simply applied using the same coefficient to all terms of the vector of the nodal velocities, which means that the term $\mathbf{C}\mathbf{v}$ has been replaced by $c\,\mathbf{v}$ in Eq. (3) and factor c is estimated in the calculation of the first eigenvalue of the system.

Another approach which seems to be new is proposed in the paper. It does not need to estimate eigenvalue calculations which can be difficult in most cases. Let us consider system (2) with the diagonalised mass matrix (although the idea is also valid for the consistent mass matrix) written in the form of separate equations width an additional dumping term D_i

$$M_i\,\dot{v}_i = Q_i + D_i \quad \text{with } D_i = -\alpha_i\,Q_i \quad (4)$$

where $\alpha_i = \alpha \in (0, 1)$ if $v_i\,Q_i > 0$ or $\alpha_i = 0$ otherwise, $Q_i = F_i - R_i$.

3. Examples

Two examples are shown to demonstrate the application of the relaxation approach in MPM: a simple problem of tension of a linearly elastic bar in the case of small deformation and the problem of slope created due to displacement of a retaining wall.

3.1. Elongation of elastic bar

The problem of elongation of a linearly elastic bar is considered. It is assumed that the left end of the bar is fixed while the point force P_0 is applied to its right end. In the dynamic analysis, the force is applied according to formula $P(t) = P_0 H(t)$ where $H(t)$ denotes the Heaviside function.

Calculations have been made with the following data: Young's modulus, $E = 1$; the cross-section area, $A = 1$; mass density, $\varrho = 1$; the length of the bar, $l = 10$; the force magnitude, $P_0 = 0.001$. In the dynamic relaxation procedure, the following values of the dumping parameter c have been applied: $\pi/10 \approx 0.31$, 0.45 and 0.70. The first of these three values follows from the eigenvalue analysis: $c = 2\,\omega_1 = 2\,\pi/2\,\sqrt{EA/(\varrho l^2)}$. The solution has been found using four Hermitian (C^1) elements and 12 material points. The results are shown in Fig. 1.

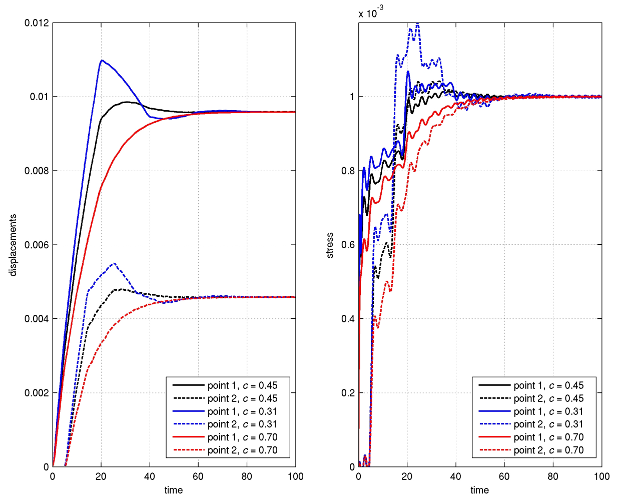

Figure 1: Displacements and stresses as functions of time for three values of the dumping parameter.

The displacements of two material points located $\frac{11}{24}l$ and $\frac{23}{24}l$ from the left bar end are shown on the left diagram in the figure for different values of the dumping coefficient. The displacements tends to the exact values: 0.0045833 and 0.0095833, respectively. The values of the stresses calculated for the same two material points are shown on the right diagram and tends with time to the exact value of P_0/A. It follows from the figures that the quasi-static solution can be reached for time $t = 50$ when coefficient c is equal to 0.45.

3.2. Slope problem

The problem of motion of a granular material retained by a wall moving slowly outwardly is considered. The plane strain problem is analysed. The initial dimensions of the cross-section of the region occupied by the material are: height of 3 m, width of 6 m. It is assumed that the wall moves with the velocity 5 cm/s, and the material is non-cohesive with the angle of internal friction 35° described by an elastic–viscoplastic constitutive model with the Drucker–Prager yield condition and a non-associated flow rule.

Calculations have been made with parameter $\alpha = 0.2$ defined in Eq. (4). Four stages of the deformation process are shown in Fig. 2. The obtained angle of the slope inclination is slightly smaller than the angle of internal friction. It should be noticed that solution of the problem by the conventional finite element method is not possible because of large distortions of the granular material in the region close to the wall. When the method is formulated in the purely Lagrangian format, a frequent re-meshing is needed in this region. Contact elements are also needed to handle the self-contact problem. On the other hand, when the method is formulated in the purely Lagrangian format, tracking the position of the free surface need introduction of additional tools, e.g. special markers. The computational time needed for the analysis (7750 elements, 22832 material points) has been about 10 hours when used Intel Core 2 Quad CPU Q6600, 2.40 GHz.

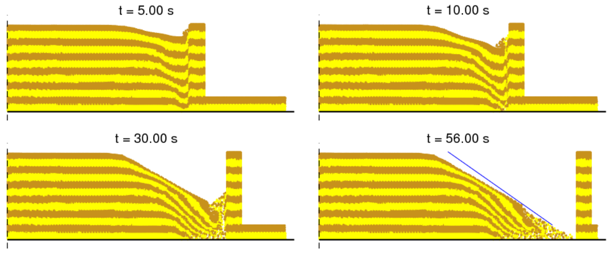

Figure 2: Several phases of the deformation process of the granular material

4. Conclusions

The dynamic relaxation technique has been applied to solve quasi-static problems by the material point method (MPM). Such an approach allows to use the dynamic formulation of MPM to solve also equilibrium problems by simple modification of an existing dynamic code. This also allows to avoid implementing MPM in quasi-static computer code which is more difficult than in the dynamic case due to necessity of constructing the stiffness matrix. The dynamic relaxation method has been demonstrated in the paper by solving one- and two-dimensional problems. The obtained results have shown usefulness of the method. It should be noticed that MPM gives less accurate results related to the stress field than FEM in the case when the latter method can handle the problem. The proposed approach is more time consuming compared to FEM. However, an application of the relaxation method to MPM with the diagonalised mass matrix in the case of 3-dimensional problems seems to be more competitive comparing to the quasi-static approach to FEM.

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