Numerical resolution of Burgers equation with moving boundary in ALE formulation : Study of coupling algorithms

T. E. Rajaomazava*, M. Benaouicha **and J.-A. Astolfi+
Research Institute of French Naval Academy (IRENav)/ENGEP.
CC600, 29240 Brest Cedex 09. France
*emerry.rajaomazava@ecole-navale.fr
**mustapha.benaouicha@ecole-navale.fr
+jacques-andre.astolfi@ecole-navale.fr

Abstract

The paper presents a study of numerical staggered schemes influence for solving nonlinear coupled problems. Arbitrary Lagrangian-Eulerian (ALE) formulation is used for Burgers equation in bounded domain with moving interface wich is described as a mass-spring problem. It illustrates a Piston motion who is an academic problem of fluid-structure interaction. A quasi-Newton iterative procedure is applied to linearize Burgers equation. The implicit and explicit coupling algorithms are studied with interface equation solved at outside then inside of Newton iterative procedure (referred as implicit-outer, implicit-inner, explicit-outer and explicit-inner respectively). It is shown that the explicit-outer algorithm requires less computing time. However, the energy conservation condition at the interface is not verified and the stability is conditioned with small time steps. The inner-explicit algorithm requires more iterations to converge than explicit-outer but energy conservation at the interface is partially verified without any convergence condition. The implicit schemes increase the global number of iterations. However, energy conservation condition at the interface is verified, and the corresponding error is controlled. In addition, the time steps can be increased while keeping the stability of the scheme.

Keywords: Nonlinear coupling problem, ALE formulation, staggered algorithm, Burgers equation

1. Introduction

Fluid-structure interaction problems occur when the fluid loading greatly affects the structure dynamics and thus, the structure displacement locally affects the fluid flow. Therefore both the structure field equations and the fluid field equations must be solved interactively. This generally gives rise to nonlinear coupled problem. The aim of this paper concerns the analysis of the linearization procedure of such problems when they are numerically solved by using the staggered algorihms. Burgers equation in bounded domain [1], coupled with a moving interface is studied. The dynamics of the interface is described as a mass-spring problem. It illustrates a Piston motion who is an academic problem of fluid-structure interaction.

2. Problem formulation and numerical approximation

Burgers equation is given in ALE formulation as :

$$\left\{\begin{array}{rlll} \rho\left[\frac{\partial u}{\partial t}+(u-w)\frac{\partial u}{\partial x}\right]= & \mu\frac{\partial^2 u}{\partial x^2} & \text{in} & \Omega & (a)\\ u= & 0 & \text{in} & \partial\Omega\backslash\Gamma & (b)\\ u= & \frac{dX}{dt} & \text{in} & \Gamma & (c)\\ u_{|t=0}= & u_0 & \text{in} & \Omega & (d)\end{array}\right. \quad (1)$$

where w is the domain displacement velocity, μ dynamic viscosity, ρ mass density, Γ the moving interface and Ω the computation domain. X describes the position of moving interface. It is given by the following equation :

$$\left\{\begin{array}{rl} \ddot{X}+\omega^2 X = & 2\frac{\mu}{m}\frac{\partial u}{\partial x}\mid_{x=X}\\ X(0) = & X_0\\ \dot{X}(0) = & \dot{X}_0\end{array}\right. \quad (2)$$

where ω and m are respectively the own pulsation and the mass of the interface. The equations (1) and (2) are coupled via the cinematic boundary condition (1-c) on Γ and the right side of the equation (2) who represents the fluid dynamic influence (fluid loading) on the interface.

A quadratic finite elements and implicit Euler scheme are used respectively for space and time discretization of Burgers equation. Quasi-Newton iterative procedure is applied to linearize equation (1). Newmark scheme [2] is used for interface equation time discretization. The domain displacement velocity w is obtained by solving the following Laplace equation :

$$\left\{\begin{array}{llll} \Delta w & =0 & in & \Omega\\ w & =\dot{X} & in & \Gamma\\ w & =0 & in & \partial\Omega\backslash\Gamma\end{array}\right. \quad\Rightarrow\quad w(x)=\frac{\dot{X}}{X}x \quad (3)$$

3. Coupling schemes

The staggered algorithm is used to solve the coupled problem (1-2). A comparative study is achieved for different coupling schemes : explicit one with interface equation solved outside then inside the Newton iterative procedure (referred as explicit-outer and explicit-inner respectively) and implicit one, by imposing energy conservation condition at the interface, with interface equation solved outside then inside the Newton iterative procedure (referred as implicit-outer and implicit-inner respectively). They are expressed, on one time step, as follows :

Explicit-outer

1 - Solve eq. (1) until convergence of Newton procedure
2 - Solve interface problem (eq. 2)
3 - Solve eq. (3) and update the mesh

Explicit-inner [3]

1 - Solve eq. (1) for one Newton iteration
2 - Solve interface problem (eq. 2)
3 - Solve eq. (3) and update the mesh
4 - Repeat 1 to 3 until convergence of Newton procedure

	Energy error at the interface	Time step	Newton iterations	Coupling iterations	CPU Time
Explicit-outer	160	$10^{-3}s$	23506 (max 3)	20000	2 mn 08 s
Explicit-inner	10^{-3}	$10^{-3}s$	25570 (max 4)	25570	2 mn 12 s
Implicit-outer	10^{-7}	$10^{-3}s$	47986 (max 15)	34771	4 mn 10 s
Implicit-inner	10^{-7}	$10^{-3}s$	34782 (max 5)	34782	2 mn 12 s

Table 1: CPU time and Number of iterations

Implicit-outer

1 - Solve eq. (1) until convergence of Newton procedure
2 - Solve interface problem (eq. 2)
3 - Solve eq. (3) and update the mesh
4 - Repeat 1 to 3 until the energy conservation is verified

Implicit-inner

1 - Solve eq. (1) for one Newton iteration
2 - Solve interface problem (eq. 2)
3 - Solve eq. (3) and update the mesh
4 - Repeat 1 to 3 until convergence of Newton procedure and the energy conservation condition is verified

The convergence of Newton iteration and energy conservation condition at the interface Γ are given respectively by:

$$\|u_n^{(k)} - u_n^{(k-1)}\| < \varepsilon_1 \quad (4)$$

$$\|(\ddot{X}_n^{(l)} + \omega^2 X_n^{(l)})\,\dot{X}_n^{(l)} - 2\frac{\mu}{m}\,(\frac{\partial u_n^{(l)}}{\partial x}\cdot u_n^{(l)})_{x=X_n^{(l)}}\| < \varepsilon_2 \quad (5)$$

where the exponents k and l represent respectively to Newton and coupling iterations numbers, whereas the subscript n referes to time step number. ϵ_1 and ϵ_2 being a small positive real numbers.

4. Performances of the coupling algorithms

For small time step, all the solutions given by the different coupling schemes are very close. Differences appear as the time step increases.

4.1. Explicit shcemes

In explicit-outer algorithm, the interface problem is solved only once by time step, whereas in explicit-inner, each Newton iteration includes interface resolution and update of the grid. Figure (1) shows the number of Newton iterations in each time step for the explicite schemes.

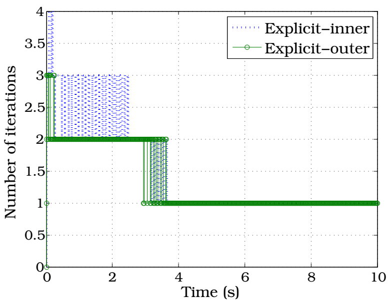

Figure 1: *Number of iterations in explicit schemes*

It is shown that the number of iterations in explicit-inner algorithm is higher compared to explicit-outer one. Less iterations are necessary for explicit-outer to converge. It is deduced that the explicit-outer algorithm is less expensive in computing time. However the energy conservation at the interface is almost verified for the explicit-inner with an error of 10^{-3}, wich is not the case of explicit-outer scheme (Table 1).

4.2. Implicit shcemes

The number of iterations in implicit-outer algorithm is represented in figure (2). Newton and coupling iterations are both increased significantly. It can be noted that the implicit-outer algorithm is 251.56% more expensive in computing time than explicit-outer one.

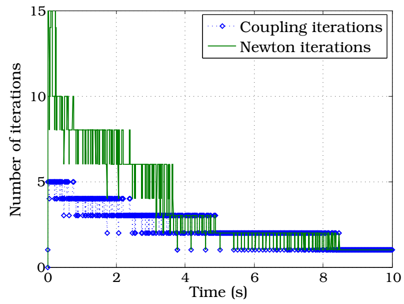

Figure 2: *Number of iterations in implicit outer*

Newton and coupling iterations are merged for implicit-inner algorithm. Figure (3) shows that the maximum of newton iterations is reduced to five. Thus, computing time is lower than in implicit-outer scheme, however it is 34.37% higher than in explicit-outer algorithm.

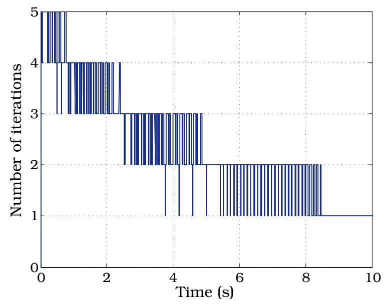

Figure 3: *Number of iterations in implicit-inner*

Explicit-inner and implicit-inner algorithms give accurate results with relatively short computing time and a verification of the energy conservation condition at the interface. It is not the case in implicit-outer scheme where the computing time is large and in explicit-outer algortihm where the energy conservation condition is not verified.

References

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