Trefftz functions for three-dimensional wave equation

Małgorzata Sokała

Faculty of Management and Computer Modelling, Kielce University of Technology, al. Tysiąclecia Państwa Polskiego 7, 25-314 Kielce, Poland, e-mail: sokala@tu.kielce.pl

Abstract

The paper presents solution of three-dimensional wave equation by using some form of Trefftz functions. These functions, wave and polywave polynomials, are obtained by inverse operation to harmonic functions, and can be used to solve the wave equations in the whole considered domain or can be used as base functions in FEM. The accuracy of the proposed method is discussed on some tested direct and inverse problems. These examples show good accuracy for an approximate solution with Trefftz functions as base functions.

Keywords: Trefftz methods, finite element methods

1. Introduction

The analytical and numerical methods for linear partial differential equations have been developing very quickly in recent years. Some of them are concerned with the wave equation. Solving a partial differential equation in regular area often leads to solution, which is linear combination of eigenfunctions [see 1,3,8]. When the considered area is more complicated the other methods are more appropriate. In paper [3] BEM was used to obtain a solution of eigenvalue problems of polygonal membranes. Work [7] contains a numerical algorithm for solution of multi-dimensional wave equations, which bases on the Houbolt finite difference (FD) scheme, the method of particular solutions (MPS) and the Fundamental Solutions Method (MFS). It allows to transform the wave equation into a Poisson-type equation with time-dependent loading.

The method presented here is a variant of Trefftz method. For these methods it is crucial to find a system of functions which satisfy identically the considered equation. An approximate solution of considered problem is represented in the form of a linear combination of Trefftz functions. Unknown coefficients of this approximation are calculated by minimization of the functional fitting the approximate solution to pre-set conditions. In the paper two methods to obtain Trefftz functions for the three-dimensional wave equation are shown. The first method is connected with a generating function, and leads to recurrent formulas for polynomials and their derivatives [4]. The second method is based on a Taylor series expansion, and uses additionally an inverse Laplace operator. This leads to other form of Trefftz wave functions which can be used as base functions in FEM. This approach was applied in [2,5] to generate Trefftz functions for the heat conduction equation. Presented work is the extension of paper [6] on the 3D wave equation. Moreover, it shows how to use the new form of Trefftz functions as the base functions in modified FEM.

2. Trefftz functions for homogenous wave equation

Let us consider the non-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial t^2} \tag{1}$$

The one of concepts of generating Trefftz functions for equation (1) is based on the Taylor series expansion for function $u(x,y,z,t)$satisfying (1), namely

$$u(x,y,z,t)=\sum_{n=0}^{\infty}\frac{d^n u(x_0,y_0,z_0,t_0)}{n!}=\sum_{n=0}^{\infty}\frac{\left(\frac{\partial}{\partial x}(x-x_0)+\frac{\partial}{\partial y}(y-y_0)+\frac{\partial}{\partial z}(z-z_0)+\frac{\partial}{\partial t}(t-t_0)\right)^n u(x_0,y_0,z_0,t_0)}{n!} \tag{2}$$

Using (1) we transform the right hand side of (2) to eliminate $\frac{\partial^2 u}{\partial t^2}$ and then group other partial derivatives in a specific way. This leads to a sum whose coefficients are functions satisfying equation (1). These functions are constructed by inverting the Laplace operator of harmonic functions.

The functions obtained by the method presented above can be transformed to other coordinates system by the proper, commonly known substitutions.

3. Solution of the wave equation

A general solution of a non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+Q(x,y,z,t)=\frac{\partial^2 u}{\partial t^2} \tag{3}$$

is given by

$$u(x,y,z,t)=L^{-1}(0)+L^{-1}(Q)$$
$$L=\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2} \tag{4}$$

A general solution $L^{-1}(0)$ of a homogeneous equation is looked for as a linear combination of Trefftz functions (wave polynomials) with unknown coefficients. They are determined by the minimization of functional describing the adjustment of the approximation (in mean square sense) to the given boundary and initial conditions. The particular solution is given as a linear combination of polywave functions, obtained by acting operator L^{-1} on Trefftz wave polynomials. The formula for $L^{-1}(x^k y^l z^m t^n)$ is given in [4].

The effectiveness of the presented method is tested on simple examples describing the direct and inverse problems under three-dimensional wave equation. An unquestionable advantage of this method is the fact that Trefftz functions (wave polynomials) do not yield any restrictions for the shape of the area, so the method can also be used to more complicated shapes.

4. Trefftz functions as a base functions in FEM

Trefftz functions (wave polynomials) can be used in the construction of base functions in FEM. The general idea of such an approach is as follows. We divide the domain into subdomains (rectangles, triangles, etc.) $\Omega_j, j = 1,2,3, \dots J$, and introduce a local system of coordinates in the space-time domains $\Omega j \times < k\Delta t, (k+1)\Delta t > \ k = 0, 1, \dots, K$. In each element the approximate solution is expressed in a form of a linear combination of base functions (they depend on both space and time variables)

$$w_j(x,y,z,t) \approx u_j(x,y,z,t) = \sum_{n=1}^{N} a_{jn} v_n(\bar{x},\bar{y},\bar{z},\bar{t}) \qquad (5)$$

Unknown coefficients in these combinations are sought by fitting the approximate solutions to the initial and boundary conditions, and by fitting solutions and their normal derivatives between the elements. Because of the Runge phenomenon of waving of polynomials at the boundary, the division is made in such a way that the relatively high degree of approximation in the element was kept, and that the matrices which form the system of equations were not badly conditioned**.** In a local system of coordinates we perform the rescaling of this system in such a way that the space-time coordinates in the element do not exceed 1.

5. Numerical example

Let us consider the non-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

$$(x,y,z) \in (0,1) \times (0,1) \times (0,1), t > 0 \qquad (6)$$

together with the conditions

$$u(0,y,z,t) = u(1,y,z,t) = u(x,0,z,t) = u(x,1,z,t) = u(x,y,0,t) = u(x,y,1,t) = 0$$

$$u(x,y,z,0) = xy(x-1)(y-1)(z-1) \quad \frac{\partial u}{\partial t}(x,y,0) = 0 \qquad (7)$$

A linear combination of Trefftz functions is an approximate solution of the problem (6-7), the result obtained for $z = 0$ and time $t = 0.5$ is shown on Fig. 1and Fig. 2.

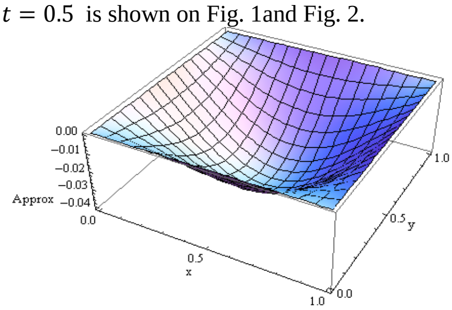

Figure 1: Approximate solution for z = 0 for time t = 0.5

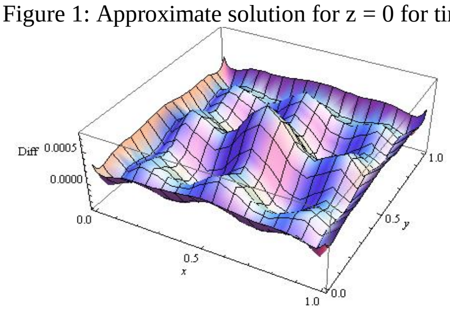

Figure 2: Difference between the exact and approximate solution at time t = 0.5 for z = 0

Fig. 3 shows comparison the exact and the approximate solution obtained for 152 Trefftz functions (polynomials of 10 degree) in the middle point of domain for one time interval.

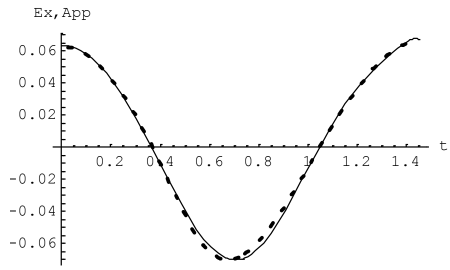

Figure 3: The exact (continuous line) and approximate solution (dashed line) at the point $x = y = z = 0.5$

As you can see the approximate solution obtained by using Trefftz function gave a satisfactory result.

The presented method is characterized by simple computational algorithm and can be easy adapted to other coordinate system. Using the Trefftz functions as a base function in FEM allows to consider problems in more complicated shapes. One of limitations of this method is Runge phenomena for high degree polynomials.

References

[1] Babuska I., Osborn J., Eigenvalue problems, *Handbook of numerical analysis. Finite element methods (part I) vol. II*, In: Ciarlet PG, Lions JL, editors,pp. 641-784 Amsterdam North Holland, 2001

[2] Ciałkowski M. Frąckowiak A., Funkcje cieplne i pokrewne w rozwiązywaniu wybranych równań mechaniki. Część I. Rozwiązywanie niektórych równań cząstkowych za pomocą operacji odwrotnych, *Współczesne Problemy Techniki. Studia i Materiały,* 3, pp.7-69, Wydawnictwo Uniwersytetu Zielonogórskiego, Zielona Góra, 2003.

[3] Heuer R., Irschik H.,A Boundary Element Method for Eigenvalue Problems of Polygonal Membranes and Plates, *Acta Mechanica* 66, pp.9-20, 1987

[4] Maciąg, A., Three-dimensional wave polynomials, *Mathematical Problems in Engineering 5*, pp. 583-598, 2005.

[5] Sokała M. "Analytical and numerical method of solving heat conduction problems with the use of heat functions and inverse operations" (in Polish), Doctoral thesis, Poznań, 2004

[6] Sokała, M., Solution of 2D non-homogenous wave equation by using polywave functions, *CAMES 16*, pp. 209-221, 2009.

[7] Young D.L., Gu M.H., Fan C.M., The time-marching method of fundamental solutions for wave equations, *Engineering Analysis with Boundary Elements* 33, pp.1411-1425, 2009.

[8] Zi-Cai Li, Tzon-Tzer Lu, Heng-Shuing Tsai, Cheng Aleksander H.D., The Trefftz method for solving eigenvalue problems, *Engineering Analysis with Boundary Elements 30*, pp. 292-308, 2006