Application of the higher order approximation provided by correction terms to the meshless and element-based error analysis

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Abstract

Paper focuses on recent developments in the a-posteriori error estimation in both the Meshless Finite Difference Method (MFDM) and the Finite Element Method (FEM). The boundary value problems of mechanics and civil engineering are analysed. The exact solution error is estimated, which arises after numerical analysis carried out. Especially, the global integral forms defined over specified subdomains are investigated here. We deal with several types of such global error estimates, e.g. hierarchical, smoothing and residual ones.
The global error evaluation was developed for the FEM, where the domain discretization is based on the elements. However, all those estimators may be also applied in the MFDM. Especially investigated here is the MFDM based reference solution, that is obtained by means of the analysis based on the appropriate higher order correction terms. Such procedure allows for obtaining superior quality error estimation for both the MFDM and FEM solution approaches. Concepts and algorithms are illustrated here with many 2D benchmark numerical tests and engineering examples.

Keywords: meshless methods, Meshless Finite Difference Method, error estimation, Finite Element Method.

1. Introduction

The paper focuses on recent developments in the a-posteriori error estimation in the Higher Order Meshless Finite Difference Method (HO MFDM, [3]) as well as in the Finite Element Method. The MFDM being the oldest meshless method, is nowadays one of the basic discrete methods of analysis of boundary value problems. In the MFDM nodes may be arbitrarily scattered, without any mesh structure imposed. Approximation of the unknown solution is expressed in terms of nodes only. It is obtained by means of the Moving Weighted Least Squares approximation method (MWLS, [2]).

We proposed ([3,4]) a new original concept of raising the approximation order, without the necessity of providing additional degrees of freedom in the simple low order MFD operator. Instead, additional higher order terms are considered, resulting from the Taylor series expansion of the simple MFD operator terms at each node of the MFD star. Beside the HO-derivatives, corrections may also contain singularity or discontinuity terms. These HO derivatives may be calculated using appropriate formulae composition, and the already known basic MFD solution. They are applied as correction terms to this MFD operator, modifying, however, only the right hand side of the MFD equations.

The improved HO solution may be applied in the a-posteriori error estimation [1] as the superior quality reference solution [3,4]. It provides the 2p-th order error estimation, where p denotes the basic approximation order considered. As opposed to the well-known hierarchic estimators [1], it does not need much computational effort.

We propose to apply such technique also in the analysis carried out by the FEM [1]. The correction terms are found using the standard FEM solution, and the Taylor series expansion of the simplest MFD operator possible to solve a given b.v. problem. They are applied in order to obtain a FEM/MFDM reference solution. It is used for estimation purposes [4].

The approach is tested on a variety of 2D benchmark tests, and selected applications of mechanics. The results of various types of estimation with the ones using the higher order reference solution are compared. The h-adaptive strategy is proposed then, based on the improved error estimation criteria for new nodes insertion.

2. Meshless Finite Difference Method

The MFDM may use any type of the formulation of the b.v. problem [2]. The basic MFDM solution approach [2] consists of several basic steps, which are listed below:

- nodes generation and modification,
  - domain partition (Voronoi tessellation and Delaunay triangulation),
  - domain topology determination,
- MFD star selection and classification,
- local MWLS approximation,
  mesh generation for the numerical integration (for global formulations only),
- generation of MFD operators,
- MFD discretization of boundary conditions,
- generation and solution of MFD equations,
- postprocessing by the MWLS.

3. Higher order approximation based on correction terms

In the standard MFDM, differential operators are replaced by finite difference ones, with a prescribed approximation order. There are several techniques that may be used for raising this order. The standard approach relies on introducing additional nodes (or degrees of freedom) into a simple MFD star for raising the approximation order. The concept of the Higher Order Approximation (HOA, [3]), used here, is based on consideration of additional terms in the Taylor expansion of the sought function at nodes of each MFD star. Those terms may consist of HO derivatives as well as discontinuity and/or singularity terms. They all are used here as correction terms to the standard MFD operator. In the solution approach, the correction terms allow for using the same standard order MFD operator, and modifying only the right hand side of the MFD equations. It is worth stressing that the final MFD solution suffers only from a truncation error of the Taylor series expansion.

Among many applications where the higher order correction terms may be used, discussed is mainly here an effective a-posteriori estimation of the solution itself and residual errors [2,4] as well. Local and global (integral form) estimations may use the HO correction terms to obtain a high quality reference solution. Especially investigated are here well-known global estimators initially designed for the FEM analysis [1,3] and adapted now to the MFDM solution approach.

4. A-posteriori error estimation

The improved HO solution may be applied in a-posteriori error estimation [1] as the superior quality reference solution. Reference solution is required in all types of estimators instead of the exact solution, which is usually unknown.

The HO reference solution provides error estimation of the 2*p*-th order, rather than *p*+1 provided by the existing error estimators, where *p* denotes the basic approximation order considered. As opposed to the other well-known estimators [1], it does not need so much computational effort. It is compared here with several types of estimators, namely the smoothing, hierarchic and residual ones.

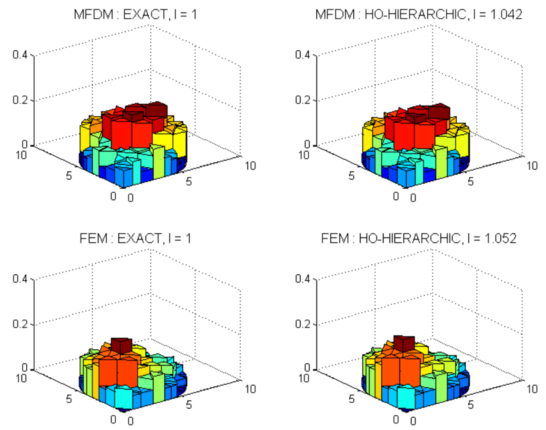

Figure 1: Results of the error estimation for both the MFDM and FEM solutions obtained by means of the higher order hierarchical estimator.

In the case of smoothing estimators, one may use correction terms explicitly for improved HO approximation of the derivatives, since smoothing is built into the MWLS approximation and HO derivatives composition. The other concept, common for the hierarchic estimators, deals with an additional solution of the MFD equations, yielding high quality HO results. An improved estimation of the residual error may also be applied in the estimators of residual type.

Moreover, adaptation, mostly in the *h*-sense, is considered here. Nodes generation criterion may be based e.g. on an improved estimation of the residual error. Those criteria, combined with some others, e.g. smoothness ones, allow for the optimal choice of node concentration zones, where either the solution or the right hand side of the differential equation exhibits large gradients. Moreover, defined and tested are several new global error indicators, possibly more sensitive to nodes irregularity than the classic integral ones. They are applied in convergence estimation of both the solution and residuals.

5. Numerical examples

Presented are (Fig.1) exemplary results for the typical 2D Poisson problem with analytical solution exhibiting large gradients. In the first column, presented are the exact errors of the MFDM (top) and FEM (bottom) solutions, evaluated on the local subdomains (elements in the FEM analysis and Delaunay triangles in the MFDM analysis). In the second columns, shown are their estimations, both of the HO-hierarchical type. Very good agreement may be observed, when both shape and effectivity index i are taken into account.

6. Final remarks

A simple and effective way of solution error estimation was proposed and tested here. It uses the concept of the higher order reference solution which may be applied in both the Meshless Finite Difference Method and in the Finite Element Method. It is based on additional correction terms resulting from the Taylor series expansion of the unknown function in the MFDM analysis. HO estimates were compared to those which are commonly applied in other discrete methods, especially in the FEM. This includes hierarchical, smoothing and residual estimators. All of them need a high quality reference solution as an equivalent of the unknown analytical solution. The HO MFD solution provides much better results for less computational effort, when compared to the high costs of other hierarchic estimators. Series of preliminary 1D and 2D benchmark tests, and several engineering problems solved, demonstrated the potential quality and power of these error estimation concepts. However, many more tests are needed, especially non-linear and 3D ones. Several chosen large engineering applications are planned as well.

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