MLPG Formulation of the Multipoint Meshless FDM

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Abstract

Discussed is here the multipoint meshless finite difference method (MMFDM) following the original Collatz [2] multipoint concept, and the essential ideas of the meshless FDM [3]. The Collatz approach was based on interpolation, regular meshes and the local formulation of b.v. problems. On the other hand in the MFDM we deal with the moving weighted least squares (MWLS) approximation, arbitrarily irregular cloud of nodes, and arbitrary global or local formulations of b.v. problems analysed. The new multipoint method [4,5,6] improves the FD solution without increasing the number of nodes in the domain. Two versions of the multipoint MFDM, namely general and specific are considered. Further extension of the multipoint MFDM is developed here. It allows for analysis of b.v. problems posed in the meshless local Petrov-Galerkin (MLPG) formulations. Several versions of the multipoint method in MLPG formulation are proposed and examined, especially the MLPG5 one, reducing amount of calculations involved. Results of tests done for the MLPG formulations, and their comparison with those obtained for the local or global ones are encouraging. Further research is planned.

Keywords: meshless finite difference method, multipoint approach, higher order approximation, weak formulations, MLPG

1. Introduction

The multipoint meshless finite difference method (MMFDM) [4, 6] has been recently developed by the authors for analysis of b.v. problems posed in the local or global formulation. It is based on the classical, original multipoint FD method proposed more then fifty years ago by L. Collatz [2], and the meshless finite difference method [3]. Presented is here an extension of the MMFD method mentioned above to the meshless local Petrov-Galerkin (MLPG) formulation [1].

The concept of the multipoint approach is based on raising the order of approximation of the unknown function by introducing additional degrees of freedom at all star nodes, using a combination of values of searched function together with a combination of the right hand side values of the considered differential equation (which are known values) or other chosen operator, evaluated in these nodes. This improves the quality of solution of boundary value problems analysed by means of the FDM without increasing the number of nodes in the mesh.

The multipoint meshless FD method is based on the moving weighted least squares MWLS approximation technique [3] instead of using the polynomial interpolation, proposed by Collatz [2]. Moreover unstructured arbitrarily irregular cloud of nodes may be used rather then the regular mesh. Besides b.v. problems posed in the local formulation (Collatz) also these, given in various global forms, may be analysed by means of the MMFDM [5].

A further extension of the multipoint meshless FDM is considered here. The method is generalized for use of the meshless local Petrov-Galerkin (MLPG) formulations, introduced by Atluri [1], especially the MLPG5.

2. Multipoint meshless FDM

Developed are two basic cases of the MMFDM approach: general and specific ones [4, 6]. Application of the specific approach is mainly restricted to the linear b.v. problems. The general formulation is more complex but it may be used for all types of b.v. problems (e.g. for non-linear ones). In the specific formulation, the additional degrees of freedom are known (like right-hand side of the eqs). In the general version of the multipoint approach, however, additional degrees of freedom are sought.

Let us consider the local (strong) formulations of boundary value problems for the n-th order ODE (PDE) $Lu = f$ with b.c. $Gu = g$, where $\mathcal{L}, \mathcal{G}$ are differential operators; or an equivalent global (weak) formulation involving integral

In the FDM solution approach, the classical difference operator would be presented in the following form

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_j c_{ij} u_j = f_i \quad \Rightarrow \quad Lu_i = f_i . \qquad (1)$$

In the multipoint formulation, the MFDM difference operator Lu is obtained by the Taylor series expansion of unknown function u, including higher order derivatives, and using additional degrees of freedom at nodes. For this purpose one may apply e.g. combination of the right hand side values f_i of the given equation at any node at each MFD star using arbitrarily distributed clouds of nodes:

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_j c_{ij} u_j = \sum_j \alpha_{ij} f_j \quad \Rightarrow \quad Lu_i = Mf_i \qquad (2)$$

It is the basic formula for the multipoint specific formulation. Here, j – number of a node in a selected FD star, Mf_i – a combination of the equation right hand side values, f_i – may present value of the whole operator $\mathcal{L}u_i$ or a combination with its derivatives. In general L may be either referred to the left side of differential eqs or to the integrand in the global formulation of b.v. problem, and to the boundary conditions.

In the general multipoint method, a specific derivative $u_i^{(k)}$ (a part of the whole operator $\mathcal{L}u_i$ only) is used as additional d.o.f. instead of the right hand side of the given differential equation

$$\sum_j c_{ij} u_j = \sum_j \alpha_{ij} u_j^{(k)} \; . \tag{3}$$

Its main drawback lies in generation rather global then local relations between the additional (unknown) d.o.f. and the basic ones. However, the required number of such relations may be reduced to N for N-dimensional b.v. problem.

Besides development of the MMFDM for analysis of b.v. problems given in the local (strong) formulation, the multipoint method was also extended to the global formulations. In particular application of MLPG5 approach is considered here.

3. The multipoint MFDM approach based on the MLPG5 formulation

The global formulation may be posed in the domain Ω as a variational principle

$$b(u, v) = l(v), \qquad \forall \; v \in V \; , \tag{4}$$

where b – bilinear functional dependent on the test function v and solution u of the considered b.v. problem, V is the space of test functions, l is a linear form dependent on v.

Assuming a trial function u locally defined on each subdomain Ω_i within the domain Ω one may obtain a global-local formulation of the Petrov-Galerkin type. The test function v may be defined in various ways. In particular one may assume it as also given locally in each subdomain Ω_i. Usually the test function is assumed to be equal to zero elsewhere, though it may be defined in many other ways.

MLPG5 formulation – Heaviside test function

In this formulation [1] the Heaviside type test function is assumed. In each subdomain Ω_i around a node P_i $i = 1, 2, \ldots N$ in the given domain Ω (e.g. in each Voronoi polygon in 2D) the test function is equal to one ($v = 1$ in Ω_i) and is assumed to be zero otherwise.

Hence any derivative of v is also equal to zero in the whole domain Ω. Therefore, relevant expressions in the functional $b(u, v)$ and in $l(v)$ vanish, reducing in this way amount of calculations involved.

In a 1D example of the beam deflections in each subdomain we have

$$u'' = f \quad \rightarrow \quad \int_0^L (u'v' + f \cdot v)\, dx = u'v\big|_0^L \quad \rightarrow \quad \int_{\Omega_i} f \, dx = u'\big|_{\Omega_i} \; ,$$

where $u(x) \in \mathrm{H}^1$, $v(x) \in \mathrm{H}^1$.

The multipoint MFDM is applied to approximation of derivatives of trial functions on the boundaries of subdomains.

After numerical integration of the left side of the Petrov-Galerkin equation by means of the Gaussian quadrature the following system of discrete simultaneous equations is received

$$J \sum_{k=1}^{g} w_k \cdot f_k \approx \sum_{j=1}^{m} \left(C_j u_j + \alpha_j f_j \right) . \tag{5}$$

Here f_k – right-hand side value in the Gaussian point P_k, w_k – corresponding weighting factor, J – Jacobian of subdomain Ω_i, C_j, α_j – coefficients of the multipoint method.

Linear test function inside each subdomain Ω_i

A linear, different from zero test function may be also chosen on each patch of the Delaunay triangles (in 2D) having one common node P_i, $i = 1,2,\ldots N$. In each triangle we have $v = ax + by + c$ in triangle $\Omega_s \subset \Omega_i$, hence $v_x' = a$, $v_y' = b$.

Various chosen 1D and 2D variants of the new multipoint method were tested. The results were compared with the corresponding ones obtained for the global local MMFDM formulations. The results of numerical tests done (eg. Figure 1) proved to be close enough to each other.

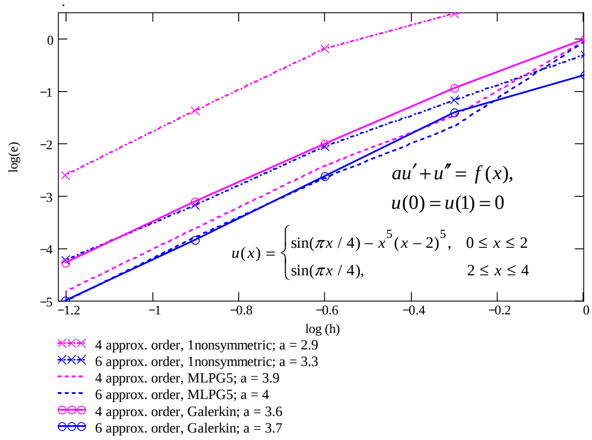

Figure 1: Comparison of the solution convergence results for different versions of multipoint method

However, clear advantage of the MLPG5 formulation relies on a significant reduction of numerical operations needed to obtain the final solution of the problems tested. Several tests of application of the new multipoint method in the boundary value problems will be presented.

4. Final remarks

The higher order approximation multipoint technique in the MFDM was considered. It is based on arbitrary irregular meshes, MWLS approximation and the global, local or global-local formulations of b.v. problems.

The recent work presented here is focused on the meshless local Petrov-Galerkin (MLPG) formulations of the multipoint meshless FDM, especially on the MLPG5. Numerical results obtained for different global, local and mixed global-local formulations are close enough. Variety of tests done confirm the observation that all variants of the multipoint versions tested, provide reasonable results.

Further development of the multipoint meshless FDM approach based on the MLPG formulation is planed.

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