Topological sensitivity derivative and its application to analysis of nucleation and growth of inclusions and voids in structures

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Abstract

The general form of topological sensitivity derivative for the strain and displacement functional with respect to volume or area of void or inclusion of different shape introduced in a structure, is presented. It is derived from shape sensitivity for vanishing size parameter and expressed in terms of primary and adjoint states. Also special case of the strain energy, when the sensitivity can be expressed only in terms of the primary fields, is considered. The results can be useful in analysis of nucleation and growth of inclusions and voids and in optimal design procedures by selecting positions, shape and orientation of these modification.

Keywords: topological sensitivity derivative, voids, inclusions, plates

1. Introduction

The present paper is devoted to determination of the topological sensitivity derivative of an arbitrary strain and displacement functional with respect to the volume (area) of inclusion or void introduced in the structure subjected to boundary tractions, body forces and displacements. For an infinitesimal modification the derivative has finite value and can be applied in analysis of nucleation and growth of inclusions or voids and in optimal design procedures. This derivation generalizes and extends previous results presented in [1] , [2] and [8].

2. Topological sensitivity derivative with respect to introduction of voids and inclusions – 3D case

Consider now an elastic structure, which occupies the domain $V \subset R^3$, with the boundary $S = S_u \cup S_T$, Fig. 1. The structure is loaded by tractions $\mathbf{T}^0$ on the boundary S_T and by body forces $\mathbf{p}_1^0$, $\mathbf{p}_2^0$ applied respectively in the structure domains V_ξ and V_0. The stress and strain states occurring in the structure domain can be presented in the vector form, namely, respectively $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T$ and $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]^T$. Moreover, displacements $\mathbf{u} = \mathbf{u}^0$ are specified on the boundary S_u .

2.1. *Definition of topological sensitivity derivative*

The topological derivative of the functional *G* with respect to the hole or inclusion volume is defined as follows (cf. [1])

$$T_{,V_0}^G(\mathbf{x}) = \lim_{V_0 \to 0} \frac{G(V - V_0) - G(V)}{V_0} = \lim_{\xi \to 0} \frac{G(V_\xi) - G(V)}{V_0^{(fix)} \xi^3}, \qquad (1)$$

where **x** (**x** ∈ *V*) is an arbitrary position in the plate domain, in which the derivative is specified, V_0, $V_0^{(fix)}$ denote respectively the domain of void (inclusion) and the reference domain, ξ is the expansion parameter, which specifies the size of the modification and $V_\xi = V - V_0$.

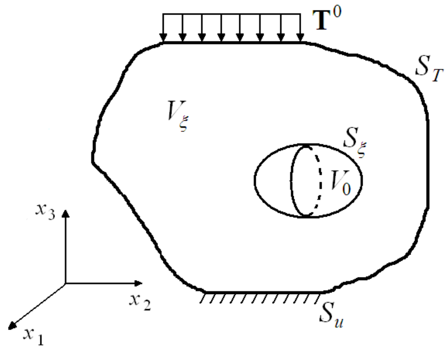

Figure 1: Introduction of inclusion or void – 3D case

2.2. *Topological sensitivity derivative with respect to introduction of inclusion*

Let us consider functional of the form

$$G = \int_{V_\xi} F_1(\boldsymbol{\varepsilon}_1)\, dV + \int_{V_0} F_2(\boldsymbol{\varepsilon}_2)\, dV + \int_{V_\xi} f_1(\mathbf{u}_1)\, dV + \int_{V_\xi} f_2(\mathbf{u}_2)\, dV + \int_{S_T} g(\mathbf{u}_1)\, dS_T, \qquad (2)$$

where $\boldsymbol{\varepsilon}_1$, $\mathbf{u}_1$ and $\boldsymbol{\varepsilon}_2$, $\mathbf{u}_2$ are the strain and displacement fields respectively in V_ξ and V_0 domains. Here, the variational approach is applied to specify the sensitivity derivative. Now, the first variation of functional (2) in the expansion process of inclusion located at the material point **x** is expressed as follows

$$\delta G = \int_{V_\xi} \frac{\partial F_1}{\partial \boldsymbol{\varepsilon}_1} \cdot \delta\boldsymbol{\varepsilon}_1 \, dV + \int_{V_0} \frac{\partial F_2}{\partial \boldsymbol{\varepsilon}_2} \cdot \delta\boldsymbol{\varepsilon}_2 \, dV + \\ + \int_{V_\xi} \frac{\partial f_1}{\partial \mathbf{u}_1} \cdot \delta\mathbf{u}_1 \, dV + \int_{V_0} \frac{\partial f_2}{\partial \mathbf{u}_2} \cdot \delta\mathbf{u}_2 \, dV + \\ + \int_{S_\xi} ([F]+[f]) n_k \delta\varphi_k dS_\xi + \int_{S_T} \frac{\partial g}{\partial \mathbf{u}_1} \cdot \delta\mathbf{u}_1 dS_T, \tag{3}$$

where $\mathbf{n} = [n_1, n_2, n_3]^T$ is the unit vector normal to the interface Γ_ξ, $\delta\boldsymbol{\varphi} = [\delta\varphi_1, \delta\varphi_2, \delta\varphi_3]^T$ is the interface transformation vector, $[F] = F_1 - F_2$ denotes "jump" of the quantity F on the interface S_ξ and $(\cdot)$ denotes the scalar product. Following the previous derivation for plates cf. [2] and the general methodology of sensitivity analysis cf. [3], the variations of state fields can be eliminated by introducing an adjoint plate structure of the same form, as the primary plate, but with induced initial stress and body force fields, namely

$$\boldsymbol{\sigma}_1^{ai} = \frac{\partial F_1}{\partial \boldsymbol{\varepsilon}_1}, \qquad \mathbf{p}_1^{a0} = \frac{\partial f_1}{\partial \mathbf{u}_1} \text{ in } V_\xi, \\ \boldsymbol{\sigma}_2^{ai} = \frac{\partial F_2}{\partial \boldsymbol{\varepsilon}_2}, \qquad \mathbf{p}_2^{a0} = \frac{\partial f_2}{\partial \mathbf{u}_2} \text{ in } V_0, \tag{4}$$

and satisfying the following boundary conditions

$$\mathbf{T}^{a0} = \frac{\partial g}{\partial \mathbf{u}_1} \quad \text{on } S_T, \qquad \mathbf{u}_1^{a0} = \mathbf{0} \quad \text{on } S_u, \tag{5}$$

where $\boldsymbol{\sigma}^a$, $\mathbf{u}^a$, $\boldsymbol{\varepsilon}^a$ are the state fields in the adjoint structure. Taking into account continuity conditions on the interface S_ξ, namely

$$[\varepsilon_{tt}] = 0, \quad [\varepsilon_{ss}] = 0, \quad [\varepsilon_{ts}] = 0, \\ [\sigma_{nn}] = 0, \quad [\sigma_{nt}] = 0, \quad [\sigma_{ns}] = 0, \tag{6}$$

where respectively, n denotes normal and t, s tangential directions to this interface, and using virtual work equation

$$\int_{V_\xi} \boldsymbol{\sigma}_1^{ar} \cdot \delta\boldsymbol{\varepsilon}_1 \, dV + \int_{V_0} \boldsymbol{\sigma}_2^{ar} \cdot \delta\boldsymbol{\varepsilon}_2 \, dV = \\ \int_{S_T} \mathbf{T}^{a0} \cdot \delta\mathbf{u}_1 dS_T + \int_{V_\xi} \mathbf{p}_1^{a0} \cdot \delta\mathbf{u}_1 \, dV + \int_{V_0} \mathbf{p}_2^{a0} \cdot \delta\mathbf{u}_2 \, dV + \\ - \int_{S_\xi} (\sigma_{nn}^a [\varepsilon_{nn}] + 2\sigma_{nt}^a [\varepsilon_{nt}] + 2\sigma_{ns}^a [\varepsilon_{ns}]) \, \delta\varphi_n dS_\xi, \tag{7}$$

next, the complementary virtual work equation

$$\int_{V_\xi} \boldsymbol{\sigma}_1^a \cdot \delta\boldsymbol{\varepsilon}_1 \, dV + \int_{V_0} \boldsymbol{\sigma}_2^a \cdot \delta\boldsymbol{\varepsilon}_2 \, dV = \\ = \int_{V_\xi} \boldsymbol{\varepsilon}_1^a \cdot \delta\boldsymbol{\sigma}_1 \, dV + \int_{V_0} \boldsymbol{\varepsilon}_2^a \cdot \delta\boldsymbol{\sigma}_2 \, dV = \int_{S_\xi} [\mathbf{p}^0] \cdot \mathbf{u}^a \delta\varphi_n \, dS_\xi + \\ - \int_{S_\xi} ([\sigma_{tt}]\varepsilon_{tt}^a + [\sigma_{ss}]\varepsilon_{ss}^a + 2[\sigma_{ts}]\varepsilon_{ts}^a) \, \delta\varphi_n dS_\xi, \tag{8}$$

we finally obtain

$$\delta G = \int_{S_\xi} (-[\sigma_{tt}]\varepsilon_{tt}^a - [\sigma_{ss}]\varepsilon_{ss}^a - 2[\sigma_{ts}]\varepsilon_{ts}^a + \sigma_{nn}^a [\varepsilon_{nn}] + \\ + 2\sigma_{nt}^a [\varepsilon_{nt}] + 2\sigma_{ns}^a [\varepsilon_{ns}] + [\mathbf{p}^0] \cdot \mathbf{u}^a + [F] + [f]) \delta\varphi_n dS_\xi. \tag{9}$$

Let us introduce general curvilinear coordinate system, where the first of coordinates ξ attains constant value on the interface S_ξ, while next two ϕ, θ change respectively in the intervals $\langle 0; 2\pi \rangle$ and $\langle 0; \pi \rangle$. In particular case let us choose generalized spherical coordinates and assume that inclusion is of the ellipsoidal shape. The relation between coordinates with axes x_{10}, x_{20}, x_{30} coinciding with the ellipsoid semi-axes ξa, ξb, ξc, and curvilinear system ξ, ϕ, θ are

$$x_{10} = \xi a \cos\phi \sin\theta, \\ x_{20} = \xi b \sin\phi \sin\theta, \\ x_{30} = \xi c \cos\theta. \tag{10}$$

Now, using the angles φ, θ as the parameters of the ellipsoid boundary, the sensitivity (9) can be expressed as follows

$$\delta G = \int_0^{\pi}\int_0^{2\pi} (-[\sigma_{tt}]\varepsilon_{tt}^a - [\sigma_{ss}]\varepsilon_{ss}^a - 2[\sigma_{ts}]\varepsilon_{ts}^a + \\ + \sigma_{nn}^a [\varepsilon_{nn}] + 2\sigma_{nt}^a [\varepsilon_{nt}] + 2\sigma_{ns}^a [\varepsilon_{ns}] + \\ + [\mathbf{p}^0] \cdot \mathbf{u}^a + [F] + [f]) \delta\varphi_n \sqrt{H} \, d\phi d\theta, \tag{11}$$

where

$$H = \xi^4 (a^2 b^2 \sin^2\theta \cos^2\theta + b^2 c^2 \cos^2\phi \sin^4\theta + \\ + a^2 c^2 \sin^2\phi \sin^4\theta). \tag{12}$$

The unit vector normal to the boundary S_ξ can be expressed as follows

$$\mathbf{n} = \left[-\frac{bc\cos\phi \sin^2\theta}{\sqrt{H}}, -\frac{ac\sin\phi \sin^2\theta}{\sqrt{H}}, -\frac{ab\sin\theta\cos\theta}{\sqrt{H}} \right]^T. \tag{13}$$

The transformation function specifying the radial boundary evolution now takes the form

$$\boldsymbol{\varphi} = [\xi a \cos\phi \sin\theta, \ \xi b \sin\phi \sin\theta, \xi c \cos\theta]^T \tag{14}$$

and its variation is

$$\delta\boldsymbol{\varphi} = \frac{\partial \boldsymbol{\varphi}}{\partial \xi} \delta\xi = [a\cos\phi \sin\theta \delta\xi, \ b\sin\phi \sin\theta \delta\xi, c\cos\theta \delta\xi]^T. \tag{15}$$

So, in view of (12), (13) and (15) the variation of the functional G expressed by (11), can be presented in the form

$$\delta G = \frac{\partial G}{\partial \xi} \delta\xi = \xi^2 abc \int_0^{\pi}\int_0^{2\pi} ([\sigma_{tt}]\varepsilon_{tt}^a + [\sigma_{ss}]\varepsilon_{ss}^a + 2[\sigma_{ts}]\varepsilon_{ts}^a + \\ - \sigma_{nn}^a [\varepsilon_{nn}] - 2\sigma_{nt}^a [\varepsilon_{nt}] - 2\sigma_{ns}^a [\varepsilon_{ns}] + \\ - [\mathbf{p}^0] \cdot \mathbf{u}^a - [F] - [f]) \sin\theta d\phi d\theta \delta\xi, \tag{16}$$

and it vanishes for $\xi = 0$. To express the sensitivity derivative with respect to the inclusion volume $V = 4/3\pi\xi^3 abc$, the following formula is obtained using the relation (16) and the incremental form $dV = 4\pi\xi^2 ab \, d\xi$, thus

$$T_{,V_0}^G(\mathbf{x}) = \left(\frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial V} \right) \Bigg|_{\xi=0} = \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} ([\sigma_{tt}]\varepsilon_{tt}^a + [\sigma_{ss}]\varepsilon_{ss}^a + \\ + 2[\sigma_{ts}]\varepsilon_{ts}^a - \sigma_{nn}^a [\varepsilon_{nn}] - 2\sigma_{nt}^a [\varepsilon_{nt}] - 2\sigma_{ns}^a [\varepsilon_{ns}] + \\ - [\mathbf{p}^0] \cdot \mathbf{u}^a - [F] - [f]) \sin\theta d\phi d\theta. \tag{17}$$

When the three last terms in (2) disappear, $[\mathbf{p}^0] = \mathbf{0}$, while the two first terms correspond to the total strain energy U, we get

$$T^{U}_{,A_0}(\mathbf{x}) = \frac{1}{8\pi}\int_0^{2\pi}(([\sigma_{tt}]\varepsilon_{tt} + [\sigma_{ss}]\varepsilon_{ss} + 2[\sigma_{ts}]\varepsilon_{ts} + \\ -\sigma_{nn}[\varepsilon_{nn}] - 2\sigma_{nt}[\varepsilon_{nt}] - 2\sigma_{ns}[\varepsilon_{ns}])d\theta. \quad (18)$$

2.3. Topological sensitivity derivative with respect to introduction of void

Taking into account that on the free boundary of the void $\sigma^a_{nn} = 0$, $\sigma^a_{nt} = 0$, $\sigma^a_{ns} = 0$ the general formula (17) now takes the form

$$T^{G}_{,V_0}(\mathbf{x}) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}(\sigma_{tt}\varepsilon^a_{tt} + \sigma_{ss}\varepsilon^a_{ss} + \\ + 2\sigma_{ts}\varepsilon^a_{ts} - \mathbf{p}^0\cdot\mathbf{u}^a - F - f)\sin\theta d\phi d\theta. \quad (19)$$

When the functional G represents the strain energy U and $\mathbf{p}^0 = \mathbf{0}$, formula (18) can be rewritten as follows

$$T^{U}_{,A_0}(\mathbf{x}) = \frac{1}{8\pi}\int_0^{2\pi}((\sigma_{tt}\varepsilon_{tt} + \sigma_{ss}\varepsilon_{ss} + 2\sigma_{ts}\varepsilon_{ts})d\theta. \quad (20)$$

3. Topological sensitivity derivative with respect to introduction of voids and inclusions in plates

Consider now an elastic plate, whose middle surface occupies the domain $A \subset R^2$, with the boundary $\Gamma = \Gamma_u \cup \Gamma_T$, Fig. 2. The plate is loaded by tractions $\mathbf{T}^0$ on the boundary Γ_T and by body forces $\mathbf{p}^0_1$, $\mathbf{p}^0_2$ applied respectively in the plate domains A_ξ and A_0. The stress and strain states occurring in the plate domain can be presented in the vector form as $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T$, $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]^T$. Moreover, displacements $\mathbf{u} = \mathbf{u}^0$ are specified on the boundary Γ_u.

3.1. Definition of topological sensitivity derivative

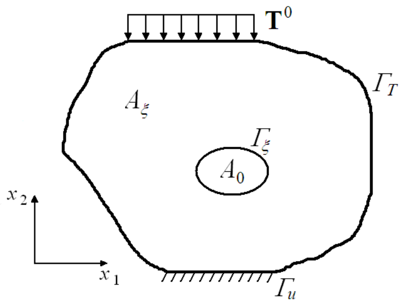

Figure 2: Introduction of inclusion or void into plate

The topological derivative of the functional *G* with respect to the hole or inclusion area is defined as follows (cf. [1])

$$T^{G}_{,A_0}(\mathbf{x}) = \lim_{A_0\to 0}\frac{G(A - A_0) - G(A)}{A_0} = \lim_{\xi\to 0}\frac{G(A_\xi) - G(A)}{A_0^{(fix)}\xi^2}, \quad (21)$$

where **x** (**x** ∈ A) is an arbitrary position in the plate domain, in which the derivative is specified, A_0, $A_0^{(fix)}$ denote respectively the domain of void (inclusion) and the reference domain, ξ is the expansion parameter, which specifies the size of the modification and $A_\xi = A - A_0$.

3.2. Topological sensitivity derivative with respect to introduction of inclusion

Let us consider functional of the form

$$G = \int_{A_\xi} F_1(\boldsymbol{\varepsilon}_1)\, dA + \int_{A_0} F_2(\boldsymbol{\varepsilon}_2)\, dA + \int_{A_\xi} f_1(\mathbf{u}_1)\, dA + \\ + \int_{A_\xi} f_2(\mathbf{u}_2)\, dA + \int_{\Gamma_T} g(\mathbf{u}_1)\, d\Gamma_T, \quad (22)$$

where $\boldsymbol{\varepsilon}_1$, $\mathbf{u}_1$ and $\boldsymbol{\varepsilon}_2$, $\mathbf{u}_2$ are the strain and displacement fields respectively in A_ξ and A_0 domains. Here, the variational approach is applied to specify the sensitivity derivative. Now, the first variation of functional (2) in the expansion process of inclusion located at the material point $\mathbf{x}$ is expressed as follows

$$\delta G = \int_{A_\xi}\frac{\partial F_1}{\partial \boldsymbol{\varepsilon}_1}\cdot\delta\boldsymbol{\varepsilon}_1\, dA + \int_{A_0}\frac{\partial F_2}{\partial \boldsymbol{\varepsilon}_2}\cdot\delta\boldsymbol{\varepsilon}_2\, dA + \\ + \int_{A_\xi}\frac{\partial f_1}{\partial \mathbf{u}_1}\cdot\delta\mathbf{u}_1\, dA + \int_{A_0}\frac{\partial f_2}{\partial \mathbf{u}_2}\cdot\delta\mathbf{u}_2\, dA + \\ + \int_{\Gamma_\xi}([F] + [f])n_k\delta\varphi_k d\Gamma_T + \int_{\Gamma_T}\frac{\partial g}{\partial \mathbf{u}_1}\cdot\delta\mathbf{u}_1 d\Gamma_T, \quad (23)$$

where $\mathbf{n} = [n_1,\, n_2]^T$ is the unit vector normal to the interface Γ_ξ, $\delta\boldsymbol{\varphi} = [\delta\varphi_1,\, \delta\varphi_2]^T$ is the interface transformation vector, $[F] = F_1 - F_2$ denotes "jump" of the quantity F on the interface Γ_ξ and $(\cdot)$ denotes the scalar product. Following the previous derivation for plates cf. [2] and the general methodology of sensitivity analysis cf. [3], the variations of state fields can be eliminated by introducing an adjoint plate structure of the same form, as the primary plate, but with induced initial stress and body force fields, namely

$$\boldsymbol{\sigma}^{ai}_1 = \frac{\partial F_1}{\partial \boldsymbol{\varepsilon}_1}, \qquad \mathbf{p}^{a0}_1 = \frac{\partial f_1}{\partial \mathbf{u}_1} \text{ in } A_\xi, \\ \boldsymbol{\sigma}^{ai}_2 = \frac{\partial F_2}{\partial \boldsymbol{\varepsilon}_2}, \qquad \mathbf{p}^{a0}_2 = \frac{\partial f_2}{\partial \mathbf{u}_2} \text{ in } A_0, \quad (24)$$

and satisfying the following boundary conditions

$$\mathbf{T}^{a0} = \frac{\partial g}{\partial \mathbf{u}_1} \text{ on } \Gamma_T, \qquad \mathbf{u}^{a0}_1 = \mathbf{0} \text{ on } \Gamma_u, \quad (25)$$

where $\boldsymbol{\sigma}^a$, $\mathbf{u}^a$, $\boldsymbol{\varepsilon}^a$ are the state fields in the adjoint structure. Taking into account continuity conditions on the interface Γ_ξ

$$[\varepsilon_{tt}] = 0, \quad [\sigma_{nn}] = 0, \quad [\sigma_{nt}] = 0, \quad (26)$$

where n, t denote respectively normal and tangential direction to this interface and using virtual work equation

$$\int_{A_\xi}\boldsymbol{\sigma}^{ar}_1\cdot\delta\boldsymbol{\varepsilon}_1\, dA + \int_{A_0}\boldsymbol{\sigma}^{ar}_2\cdot\delta\boldsymbol{\varepsilon}_2\, dA = \\ = \int_{\Gamma_T}\mathbf{T}^{a0}\cdot\delta\mathbf{u}_1 d\Gamma_T + \int_{A_\xi}\mathbf{p}^{a0}_1\cdot\delta\mathbf{u}_1\, dA + \int_{A_0}\mathbf{p}^{a0}_2\cdot\delta\mathbf{u}_2\, dA + \\ - \int_{\Gamma_\xi}(\sigma^a_{nn}[\varepsilon_{nn}] + 2\sigma^a_{nt}[\varepsilon_{nt}])\,\delta\varphi_n d\Gamma_\xi, \quad (27)$$

next, the complementary virtual work equation

$$\int_{A_\xi} \boldsymbol{\sigma}_1^a \cdot \delta\boldsymbol{\varepsilon}_1 \, dA + \int_{A_0} \boldsymbol{\sigma}_2^a \cdot \delta\boldsymbol{\varepsilon}_2 \, dA = \int_{A_\xi} \boldsymbol{\varepsilon}_1^a \cdot \delta\boldsymbol{\sigma}_1 \, dA + \int_{A_0} \boldsymbol{\varepsilon}_2^a \cdot \delta\boldsymbol{\sigma}_2 \, dA =$$
$$= \int_{\Gamma_\xi} [\mathbf{p}^0] \cdot \mathbf{u}^a \delta\varphi_n \, dA - \int_{\Gamma_\xi} [\sigma_{tt}] \varepsilon_{tt}^a \, \delta\varphi_n d\Gamma_\xi \, , \tag{28}$$

we finally obtain

$$\delta G = \int_{\Gamma_\xi} (-[\sigma_{tt}]\varepsilon_{tt}^a + \sigma_{nn}^a[\varepsilon_{nn}] + 2\sigma_{nt}^a[\varepsilon_{nt}] + +[\mathbf{p}^0]\cdot\mathbf{u}^a + [F] + [f])\delta\varphi_n d\Gamma_\xi \, . \tag{29}$$

Let us introduce general curvilinear coordinate system, where the first of coordinates η attains constant value on the interface Γ_ξ and the second θ changes in the interval $\langle 0;2\pi \rangle$. In particular case let us choose elliptical coordinates. Then, taking ξ as the design parameter, (9) takes the form

$$\frac{\partial G}{\partial \xi} = \xi ab \int_0^{2\pi} ([\sigma_{tt}]\varepsilon_{tt}^a - \sigma_{nn}^a[\varepsilon_{nn}] - 2\sigma_{nt}^a[\varepsilon_{nt}] + -[\mathbf{p}^0]\cdot\mathbf{u}^a - [F] - [f])d\theta \, , \tag{30}$$

where ξa, ξb are the lengths of ellipse semi-axes. To express the sensitivity derivative with respect to the inclusion area $A = \pi\xi^2 ab$, the following formula is obtained using the relation (10) and the incremental form $dA = 2\pi\xi ab \, d\xi$, thus

$$T_{,A_0}^G(\mathbf{x}) = \left. \left(\frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial A} \right) \right|_{\xi=0} = \frac{1}{2\pi} \int_0^{2\pi} ([\sigma_{tt}]\varepsilon_{tt}^a - \sigma_{nn}^a[\varepsilon_{nn}] + -2\sigma_{nt}^a[\varepsilon_{nt}] - [\mathbf{p}^0]\cdot\mathbf{u}^a - [F] - [f])d\theta \, . \tag{31}$$

When the three last terms in (22) disappear, $[\mathbf{p}^0] = \mathbf{0}$, while the two first terms correspond to the total strain energy U, we get

$$T_{,A_0}^U(\mathbf{x}) = \frac{1}{4\pi} \int_0^{2\pi} ([\sigma_{tt}]\varepsilon_{tt} - \sigma_{nn}[\varepsilon_{nn}] - 2\sigma_{nt}[\varepsilon_{nt}])d\theta \, . \tag{32}$$

3.3. Topological sensitivity derivative with respect to introduction of void

Taking into account that on the free boundary of the void $\sigma_{nn} = 0$, $\sigma_{nt} = 0$ the general formula (31) now takes the form

$$T_{,A_0}^G(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{tt}\varepsilon_{tt}^a - \mathbf{p}^0 \cdot \mathbf{u}^a - F - f)d\theta \, . \tag{33}$$

When the functional G represents the strain energy U and $\mathbf{p}^0 = \mathbf{0}$, formula (32) can be rewritten as follows

$$T_{,A_0}^U(\mathbf{x}) = \frac{1}{4\pi} \int_0^{2\pi} \sigma_{tt}\varepsilon_{tt} d\theta = \frac{1}{4\pi E} \int_0^{2\pi} \sigma_{tt}^2 d\theta \, , \tag{34}$$

where E denotes the Young's modulus.

4. Determination and application of topological sensitivity derivative

In order to use derived formulae for topological derivative, the stress or strain distributions on the interface should be known. In many cases, they can be analytically determined using methods of the elasticity theory (cf. [6], [7]), or they can be specified numerically.

4.1. Topological sensitivity derivative of the strain energy with respect to introduction of circular inclusion into plate

Let us consider infinite plate working in the plane state of stress and containing circular inclusion made of different material, which is introduced in a certain point **x**. We assume that Young's modulus, shear modulus and Poisson's ratio are denoted respectively for matrix by E_1, G_1, ν_1 and for inclusion by E_2, G_2, ν_2.

The displacement and stress components appearing in the matrix, in the coordinate system n,t, are (cf. [7])

$$u_n^{(1)} = \frac{\sigma_1 + \sigma_2}{8G_1 r}[(\kappa_1 - 1)r^2 + 2\gamma_1 R^2] + + \frac{\sigma_1 - \sigma_2}{8G_1 r}\left[\beta_1(\kappa_1 - 1)r^2 + 2r^2 + \frac{2\delta_1 R^4}{r^2} \right]\cos 2\theta, \tag{35}$$

$$u_t^{(1)} = -\frac{\sigma_1 - \sigma_2}{8G_1 r}\left[\beta_1(\kappa_1 - 1)r^2 + 2r^2 - \frac{2\delta \, R^4}{r^2} \right]\sin 2\theta, \tag{36}$$

and

$$\sigma_{nn}^{(1)} = \frac{\sigma_1 + \sigma_2}{2}\left(1 - \frac{\gamma_1 R^2}{r^2} \right) + + \frac{\sigma_1 - \sigma_2}{2}\left(1 - \frac{2\beta_1 R^2}{r^2} - \frac{3\delta_1 R^4}{r^4} \right)\cos 2\theta, \tag{37}$$

$$\sigma_{tt}^{(1)} = \frac{\sigma_1 + \sigma_2}{2}\left(1 + \frac{\gamma_1 R^2}{r^2} \right) - \frac{\sigma_1 - \sigma_2}{2}\left(1 - \frac{3\delta_1 R^4}{r^4} \right)\cos 2\theta, \tag{38}$$

$$\sigma_{nt}^{(1)} = -\frac{\sigma_1 - \sigma_2}{2}\left(1 + \frac{\beta_1 R^2}{r^2} + \frac{3\delta_1 R^4}{r^4} \right)\sin 2\theta, \tag{39}$$

where σ_1, σ_2 denote the principal stresses in the neighbourhood of the point $\mathbf{x}$, r,θ are the polar coordinates, R is the radius of inclusion (Fig. 3) and β_1, γ_1, δ_1 are the unknown constants.

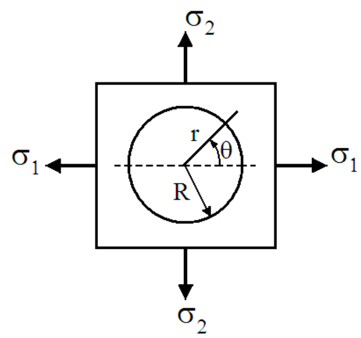

Figure 3: Introduction of circular inclusion or void into plate in plane state of stress - notation

The displacement and stress components expressed by (35)-(39), can be presented on the boundary between matrix and inclusion i.e. for $r = R$, in the form

$$\left. u_n^{(1)} \right|_{r=R} = \frac{\sigma_1 + \sigma_2}{8G_1}[(\kappa_1 - 1)R + 2\gamma_1 R] + + \frac{\sigma_1 - \sigma_2}{8G_1}[\beta_1(\kappa_1 - 1)R + 2R + 2\delta_1 R]\cos 2\theta, \tag{40}$$

$$u_t^{(1)}\Big|_{r=R} = -\frac{\sigma_1 - \sigma_2}{8G_1}[\beta_1(\kappa_1 - 1)R - 2\delta_1 R]\sin 2\theta, \tag{41}$$

$$\begin{aligned}\sigma_{nn}^{(1)}\Big|_{r=R} &= \frac{\sigma_1 + \sigma_2}{2}(1-\gamma_1) + \\ &+ \frac{\sigma_1 - \sigma_2}{2}(1 - 2\beta_1 - 3\delta_1)\cos 2\theta,\end{aligned} \tag{42}$$

$$\sigma_{tt}^{(1)}\Big|_{r=R} = \frac{\sigma_1 + \sigma_2}{2}(1-\gamma_1) - \frac{\sigma_1 - \sigma_2}{2}(1 - 3\delta_1)\cos 2\theta, \tag{43}$$

$$\sigma_{nt}^{(1)}\Big|_{r=R} = -\frac{\sigma_1 - \sigma_2}{2}(1 + \beta_1 + 3\delta_1)\sin 2\theta. \tag{44}$$

The displacement and stress components occurring in the inclusion are (cf. [7])

$$\begin{aligned}u_n^{(2)} &= \frac{\sigma_1 + \sigma_2}{8G_2}\beta_2(\kappa_2 - 1)r + \\ &+ 2\delta_2 r - \frac{\sigma_1 - \sigma_2}{4G_2}\delta_2 r\cos 2\theta,\end{aligned} \tag{45}$$

$$u_t^{(2)} = -\frac{\sigma_1 - \sigma_2}{4G_2}\delta_2 r\sin 2\theta, \tag{46}$$

$$\sigma_{nn}^{(2)} = \frac{\sigma_1 + \sigma_2}{2}\beta_2 + \frac{\sigma_1 - \sigma_2}{2}\delta_2\cos 2\theta, \tag{47}$$

$$\sigma_{tt}^{(2)} = \frac{\sigma_1 + \sigma_2}{2}\beta_2 - \frac{\sigma_1 - \sigma_2}{2}\delta_2\cos 2\theta, \tag{48}$$

$$\sigma_{nt}^{(2)} = -\frac{\sigma_1 - \sigma_2}{2}\delta_2\sin 2\theta, \tag{49}$$

where β_2, δ_2 are the unknown constants. It is easy to notice that stress state in the inclusion is homogeneous. Using the following conditions on the boundary between the matrix and the inclusion

$$\begin{aligned}&\sigma_{nn}^{(1)}\Big|_{r=R} = \sigma_{nn}^{(2)}\Big|_{r=R}, \qquad \sigma_{nt}^{(1)}\Big|_{r=R} = \sigma_{nt}^{(2)}\Big|_{r=R}, \\ &u_n^{(1)}\Big|_{r=R} = u_n^{(2)}\Big|_{r=R}, \qquad u_t^{(1)}\Big|_{r=R} = u_t^{(2)}\Big|_{r=R},\end{aligned} \tag{50}$$

the values of unknown constants can be expressed as follows

$$\begin{aligned}&\beta_1 = -\frac{2(G_2 - G_1)}{G_1 + G_2\kappa_1}, \quad \delta_1 = \frac{G_2 - G_1}{G_1 + G_2\kappa_1}, \\ &\gamma_1 = \frac{G_1(\kappa_2 - 1) - G_2(\kappa_1 - 1)}{2G_2 + G_1(\kappa_2 - 1)}, \\ &\beta_2 = \frac{G_2(\kappa_1 + 1)}{2G_2 + G_1(\kappa_2 - 1)}, \quad \delta_2 = \frac{G_2(\kappa_1 + 1)}{G_1 + G_2\kappa_1},\end{aligned} \tag{51}$$

where for plane state of stress

$$\kappa_1 = \frac{3 - \nu_1}{1 + \nu_1}, \qquad \kappa_2 = \frac{3 - \nu_2}{1 + \nu_2}. \tag{52}$$

Now, taking into account Hooke's law for matrix

$$\begin{aligned}\varepsilon_{nn}^{(1)} &= \frac{1}{E_1}(\sigma_{nn}^{(1)} - \nu_1\sigma_{tt}^{(1)}), \\ \varepsilon_{tt}^{(1)} &= \frac{1}{E_1}(\sigma_{tt}^{(1)} - \nu_1\sigma_{nn}^{(1)}), \\ \varepsilon_{nt}^{(1)} &= \frac{\sigma_{nt}^{(1)}}{2G_1},\end{aligned} \tag{53}$$

and analogous for inclusion, and next substituting (42)-(44) and (47)-(49) into (32), topological sensitivity derivative of strain energy with respect to introduction of circular inclusion, we finally obtain as follows

$$\begin{aligned}T_{,A_0}^U(\mathbf{x}) = \frac{1}{2E_1}\Bigg[&(\sigma_1 + \sigma_2)^2\frac{G_1(\kappa_2 - 1) - G_2(\kappa_1 - 1)}{2G_2 + G_1(\kappa_2 - 1)} + \\ &+ (\sigma_1 - \sigma_2)^2(G_1 - G_2)\frac{2G_1 + G_2(\kappa_1 - 1)}{(G_1 + G_2\kappa_1)^2} + \\ &+ (\sigma_1 - \sigma_2)^2(G_1 - G_2)\frac{G_2(1 + \nu_1)(\kappa_1 + 1)^2}{4(G_1 + G_2\kappa_1)^2}\Bigg].\end{aligned} \tag{54}$$

The diagram of topological derivative variation in function of ratio of Kirchhoff's moduli $\mu = G_2/G_1$ for different values of stress ratio $\zeta = \sigma_2/\sigma_1$ and in the case, when $\nu_1 = \nu_2 = 0.3$ is presented in Fig. 4. It is easy to notice that, as we expect, for $0 \le \mu < 1$ topological derivative is positive, for $\mu = 1$ equals zero, while for $\mu > 1$ is negative and when $\mu \to \infty$ attains finite value.

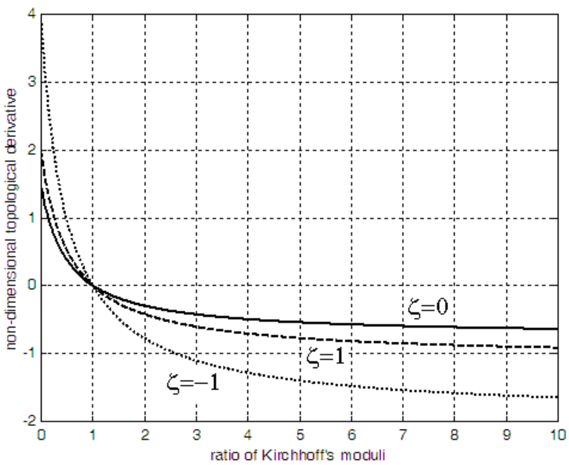

Figure 4: Non-dimensional topological derivative $\frac{E_1}{\sigma_1^2}\cdot T_{,A_0}^U$ variation in function of ratio of Kirchhoff's moduli $\mu = G_2/G_1$ for different values of stress ratio $\zeta = \sigma_2/\sigma_1$.

Similar considerations also can be done for the plane state of strain and in this case, we have that

$$\kappa_1 = 3 - 4\nu_1, \quad \kappa_2 = 3 - 4\nu_2. \tag{55}$$

4.2. Topological sensitivity derivative of the strain energy with respect to introduction of elliptical hole into plate

Let us consider infinite plate working in the plane state of stress. We assume that elliptical hole will be introduced in a certain point **x** and its major axis is oriented at angle α to the principal stress σ_1, Fig. 5.

The tangential stress distribution at the hole perimeter is specified from the elastic solution for an infinite plate uniformly loaded on the boundary (cf. [9]), thus

$$\sigma_{tt} = (\sigma_1+\sigma_2)\frac{1-\xi^2}{\xi^2-2\xi\cos 2\theta+1}+ \\ +2(\sigma_1-\sigma_2)\frac{\xi\cos 2\alpha-\cos 2(\theta+\alpha)}{\xi^2-2\xi\cos 2\theta+1}, \tag{56}$$

where

$$\xi=\frac{a-b}{a+b}=\frac{1-\eta}{1+\eta}, \qquad \eta=\frac{b}{a} \tag{57}$$

and the angle θ specifies the position on the perimeter with respect to the major ellipse axis.

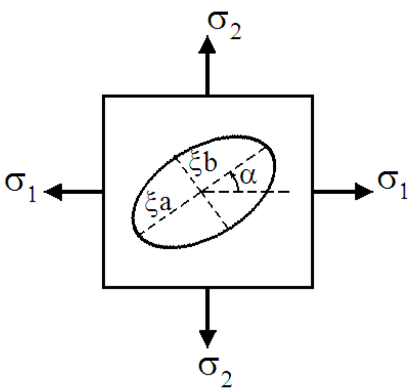

Figure 5: Introduction of elliptical hole into plate in plane state of stress – notation

The alternative form of (56) in terms of the shape parameter η is

$$\sigma_{tt} = (\sigma_1+\sigma_2)\frac{2\eta}{1+\eta^2-(1-\eta^2)\cos 2\theta}+ \\ +2(\sigma_1-\sigma_2)\frac{(1-\eta^2)\cos 2\alpha-(1+\eta)^2\cos 2(\theta+\alpha)}{1+\eta^2-(1-\eta^2)\cos 2\theta}. \tag{58}$$

The integral occurring in (34) can be presented in the form

$$\int_0^{2\pi}\sigma_{tt}^2 d\theta=\pi(\sigma_1+\sigma_2)^2\left(\eta+\frac{1}{\eta}\right)+ \\ +2\pi(\sigma_1^2-\sigma_2^2)\left(\eta-\frac{1}{\eta}\right)\cos 2\alpha+ \\ +\pi(\sigma_1-\sigma_2)^2\left(\eta+2+\frac{1}{\eta}\right). \tag{59}$$

Accounting for (59), the topological sensitivity derivative (34) can be expressed analytically as follows

$$T^U_{,A_0}(\mathbf{x},\alpha)=\frac{1}{4E}\left[(\sigma_1+\sigma_2)^2\left(\eta+\frac{1}{\eta}\right)+ \right. \\ +2(\sigma_1^2-\sigma_2^2)\left(\eta-\frac{1}{\eta}\right)\cos 2\alpha+ \\ \left. +(\sigma_1-\sigma_2)^2\left(\eta+2+\frac{1}{\eta}\right)\right]. \tag{60}$$

The diagram of topological derivative variation in function of shape parameter $\eta=b/a$ for different values of stress ratio $\zeta=\sigma_2/\sigma_1$ and orientation angle α is presented in Fig. 6.

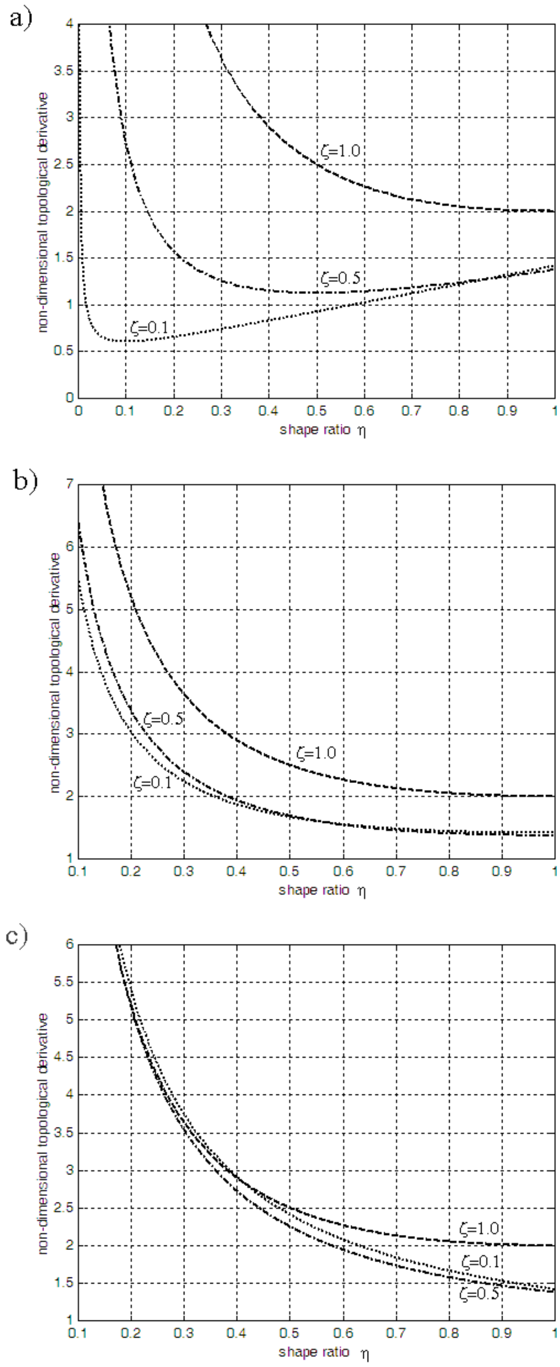

Figure 6: Non-dimensional topological derivative $\frac{E}{\sigma_1^2}\cdot T^U_{,A_0}$ variation in function of shape ratio $\eta=b/a$ for different values of stress ratio $\zeta=\sigma_2/\sigma_1$ and for orientation angles: a) $\alpha=0$; b) $\alpha=\pi/4$; c) $\alpha=\pi/2$.

4.3. Application of topological sensitivity derivative

The topological derivative expressions can be used in gradient optimization procedures by selecting position, shape and orientation of voids or inclusions and in formulation of optimality conditions. Let us consider general optimization problem defined as follows

$$\min_{p_i, i=1,2,\ldots,n} G\,, \quad \text{subject to} \quad C - C_0 \le 0\,, \tag{61}$$

where $G(p_i)$ is the objective functional expressed by (2) or (22), $C(p_i)$ denotes the global cost and C_0 is the upper bound imposed on the global cost. Introducing the Lagrangian functional

$$L(p_i, \lambda) = G + \lambda(C - C_0)\,, \qquad \lambda \ge 0 \tag{62}$$

the stationary conditions are

$$\begin{aligned} &\frac{\partial G}{\partial p_i} + \lambda \frac{\partial C}{\partial p_i} = 0, \qquad i = 1, 2, \ldots, n, \\ &\lambda(C - C_0) = 0\,. \end{aligned} \tag{63}$$

where λ is the Lagrange multiplier and $p_i, i = 1,2,\ldots,n$ are the design parameters. The sensitivity derivatives $\partial G/\partial p_i$ are specified by the formulae derived in the paper. The cost derivatives $\partial C/\partial p_i$ can easily be derived for the assumed cost function expressed in terms of design parameters. The optimal values of the parameters $p_i, i = 1,2,\ldots,n$ and the multiplier λ are determined in the incremental process of gradient optimization.

The problems of nucleation and growth of inclusions and voids can be discussed using expressions for topological sensitivity derivatives with respect to introduction of modifications, namely (17) - (20) and (31) - (34), and with respect to increase of modifications, namely (16) and (30).

The concept of topological sensitivity derivative discussed in the previous section can be also applied in material science, namely to the process of nucleation and growth of new phases, such as nucleation of a liquid from the vapour, solidification, phase transformation (such as austenite – martensite in carbon steels), recrystallization, crack generation, etc. The new phase usually, generates as a volume or plane element of finite size and next grows.

The example considered here is associated with the crack nucleation process. This problem was previously analyzed in [1], [4] and [5]. It is well known that in Griffith theory the existence of cracks of length (or diameter) $2a$ in material or structure is assumed. However, the nucleation would require infinite stress values, contrary to general observations. Fig. 7 presents the potential energy variation dependent on the crack length. We have for a wide plate loaded uniaxially, Fig. 7a

$$\Delta\Pi = \Delta\Pi_l + \Delta\Pi_s = -\frac{\pi a^2 \sigma^2}{E} + 2\gamma a\,, \tag{64}$$

where $\Delta\Pi_l$ denotes the potential energy release due to crack presence and $\Delta\Pi_s = 2\gamma a$ represents the surface energy or the dissipated energy in the process of crack growth. The critical crack length is specified by the condition

$$\frac{\partial(\Delta\Pi)}{\partial a} = 0, \quad \text{where} \quad a_c = \frac{\gamma E}{\pi\sigma^2}\,, \tag{65}$$

and for $a > a_c$ there is unstable crack growth at constant stress with associated decrease of the free energy. The constant energy transformation now corresponds to $\Delta\Pi = 0$ and

$$a_i = \frac{2\gamma E}{\pi\sigma^2}\,. \tag{66}$$

In this case, the topological transformation occurs along the path OB, Fig. 7b.

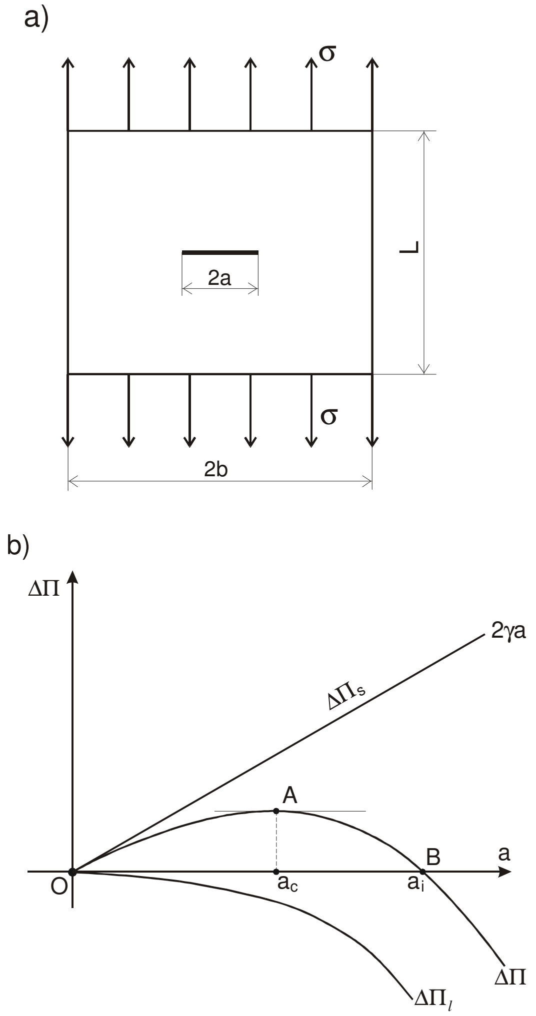

Figure 7: Plane crack under tensile stress: a) element and crack dimensions, b) potential energy and dissipation variation

5. Concluding remarks

The expressions for topological sensitivity derivative with respect to introduction of infinitesimally small inclusions and voids are derived in the paper. It was demonstrated that the topological derivative can be obtained from the shape sensitivity analysis. The presented formulae can be used in optimization procedures. They are also useful in analysis of nucleation and growth problems of inclusions and voids.

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