Analysis of composites with multiple rigid-line reinforcements by the BEM

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Abstract

This paper presents the boundary element method (BEM) for analysis of plates containing multiple rigid-line reinforcements. The accuracy of the method is verified by comparing computed stresses in the neighbourhood of a rigid fibre in an infinite plate subjected to tension with analytical solutions. The proposed approach is applied for analysis of composites reinforced by multiple rigid fibres. The influence of the distance between the fibres on stress distribution in the matrix and the effective Young modulus is analyzed.

Keywords: boundary element methods, composites, microstructures, homogenization, elasticity

1. Introduction

The aim of using fibres in composite materials is to increase strength and stiffness of structures. It is known that these two properties are difficult to improve simultaneously [3]. Stiffness depends on overall properties of the structure and strength on microstructure. If the stiffness of fibers is much greater than the matrix then the fibers can be modelled as rigid stiffeners in a deformable matrix. The examples of such natural composites are: silk, nacre and enamel [6]. The analysis of composites with many fibers requires computational methods.

Li and Ting [4] analyzed a line inclusion in an anisotropic elastic infinite plate subjected to uniform loading at infinity. They used the Stroh formalism to calculate the displacement and stress fields for the rigid and elastic inclusion. Pingle et al. [6] used the duality principle to derive rigid line inclusion solutions from the crack solutions. They computed stress fields around a rigid line inclusion and derived a compliance contribution tensor for a single and multiple line inclusions. Gorbatikh et al. [3] determined the relation between stress intensity factors at the tips of rigid line inclusions and effective compliance of the material. Liu et al. [5] applied the fast multipole BEM to analyze composites reinforced by nanotubes. They calculated effective material properties of composites modelled as three-dimensional structures having large number of degrees of freedom.

The formulation and application of the BEM for a single rigid fibre in a finite plate was presented by Fedelinski [2]. In the present work the method is extended and applied to solids with multiple rigid-line reinforcements.

2. Boundary element method for plates with rigid fibres

2.1. Boundary integral equations for a plate with fibres

Consider a plate made of homogenous, isotropic and linear-elastic material. The boundary of the plate is denoted by Γ and its domain by Ω (Fig. 1). The plate is statically loaded along the external boundary Γ by boundary tractions t_j and the domain Ω by body forces f_j. The relation between the loading of the plate and its displacements can be expressed by the Somigliana identity [1]

$$c_{ij}(x')u_j(x') + \int_\Gamma T_{ij}(x',x)u_j(x)d\Gamma(x) = \int_\Gamma U_{ij}(x',x)t_j(x)d\Gamma(x) + \int_\Omega U_{ij}(x',X)f_j(X)d\Omega(X), \quad (1)$$

where: x' is the collocation point, for which the above integral equation is applied, x is the boundary point, X is the domain point and c_{ij} is a constant, which depends on the position of the point x', U_{ij} and T_{ij} are the Kelvin fundamental solutions of elastostatics. In equations the Einstein summation convention is used and the indices for two-dimensional problems have values $i,j=1,2$.

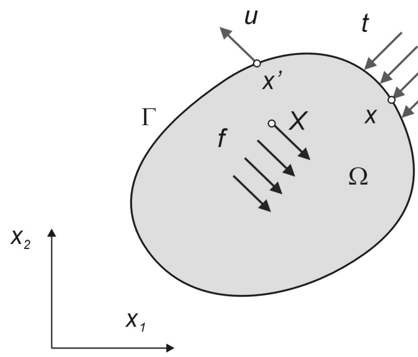

Figure 1. Elastic plate - loading and displacements

Assume that the plate is reinforced by straight, thin and rigid fibres, which are perfectly bonded to the matrix. If the plate is deformed then forces of interaction occur along the lines of attachment of fibers (Fig. 2). These forces of interaction can be treated as particular body forces acting along the lines in the domain of the body. The boundary integral equation (1) for the plate loaded by boundary tractions and forces of interaction of fibres has the form [2]

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$$c_{ij}(x')u_j(x') + \int_\Gamma T_{ij}(x',x)u_j(x)d\Gamma(x) =$$
$$\int_\Gamma U_{ij}(x',x)t_j(x)d\Gamma(x) + \sum_{n=1}^{N}\int_{\Gamma_n} U_{ij}(x',X)t_j^n(X)d\Gamma_n(X), \quad (2)$$

where: N is the number of fibres, Γ_n is the line of attachment and t_j^n are the forces of interaction.

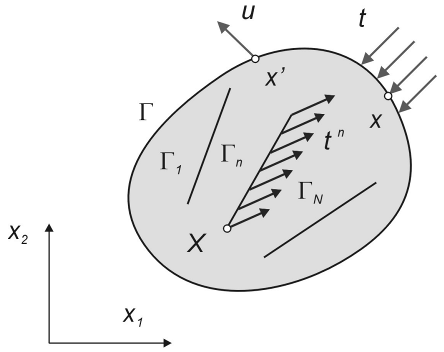


Figure 2. Elastic plate with fibres

2.2. Displacements and equations of equilibrium for stiff fibres

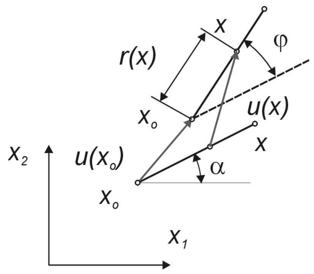


Figure 3. Displacements of the rigid fibre

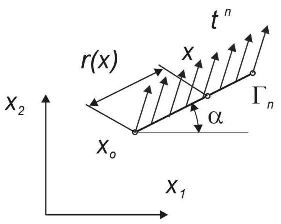


Figure 4. Forces acting on the fibre

Deformations of the matrix influence displacements of fibres. The displacement of an arbitrary point x of the fibre can be expressed by the displacements of the fibre tip x_o and the angle of rotation of the fibre φ (Fig. 3). For small angles of rotation of fibres, the components of displacements of an arbitrary point of the fibre are expressed in the form

$$u_1(x) = u_1(x_o) - \varphi r(x)\sin\alpha\,, \quad (3)$$

$$u_2(x) = u_2(x_o) + \varphi r(x)\cos\alpha\,, \quad (4)$$

where: α is the initial angle between the fibre and the axis x_1 of the global coordinate system, r is the distance between the point x and the fibre tip x_o.

The considered structure and therefore each fibre is in equilibrium. The forces acting on each fibre (Fig. 4) should satisfied the following equilibrium equations

$$\int_{\Gamma_n} t_1^n(x)d\Gamma_n(x) = 0\,, \quad (5)$$

$$\int_{\Gamma_n} t_2^n(x)d\Gamma_n(x) = 0\,, \quad (6)$$

$$\int_{\Gamma_n} [-t_1^n(x)\sin\alpha + t_2^n(x)\cos\alpha]r(x)d\Gamma_n(x) = 0\,, \quad (7)$$

The last equation (7) is the equation of moments of forces with respect of the fibre tip x_o.

2.3. Numerical implementation of the method

The boundary of the plate and the fibres are divided into boundary elements (Fig. 5). In the developed computer code 3-node quadratic boundary elements are used. Along the external boundary of the plate the variations of coordinates, displacements and tractions, and along the fibres the variations of interaction forces are interpolated. The boundary integral equations (2) are used for nodes along the external boundary and the fibres.

The displacements of fibre nodes can be expressed by the displacements of fibre tips and their angles of rotation, by using Eqs (3) and (4). These equations can written in the following matrix form

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u}_f\,, \quad (8)$$

where the matrix $\boldsymbol{u}$ contains the components of displacements of fibre nodes, the matrix $\boldsymbol{I}$ depends on the position of nodes and the matrix $\boldsymbol{u_f}$ contains components of displacements of fibre tips and their angles of rotation.

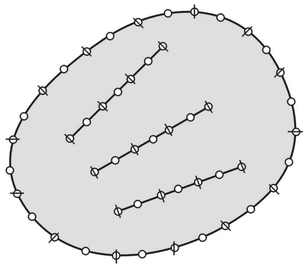


Figure 5. Discretization of the matrix and fibres by quadratic boundary elements

The equilibrium equations for fibres (5), (6) and (7) can be written in the matrix form

$$\boldsymbol{E}\boldsymbol{t}_f = 0\,, \quad (9)$$

where the matrix $\boldsymbol{E}$ depends on the position of fibre nodes and the matrix $\boldsymbol{t_f}$ contains nodal values of components of tractions in fibres. The matrix $\boldsymbol{E}$ is obtained by integration of expressions in Eqs (5), (6) i (7), by assuming quadratic variations of forces along the fibres. Because the equilibrium equations have very simple forms, the integrals are computed analytically.

Boundary integral equations (2), supplied with Eqs (8) and (9) can be written in the matrix form

$$\begin{bmatrix} \boldsymbol{H}_{ee} & 0 \\ \boldsymbol{H}_{fe} & \boldsymbol{I} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{u}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{ee} & \boldsymbol{G}_{ef} \\ \boldsymbol{G}_{fe} & \boldsymbol{G}_{ff} \\ 0 & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_e \\ \boldsymbol{t}_f \end{bmatrix}, \qquad (10)$$

where the submatrices with the index $\boldsymbol{e}$ are related to the external boundary and the submatrices denoted by the index $\boldsymbol{f}$ are related to the fibres. The submatrices $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on boundary integrals of fundamental solutions, shape functions and are integrated numerically by using the Gauss quadrature.

Next the system of algebraic equations is rearranged. The unknown quantities are on the left side of the equation and the known quantities on the right side of the equation. The first modification refers to the unknown interaction forces $\boldsymbol{t}_f$

$$\begin{bmatrix} \boldsymbol{H}_{ee} & -\boldsymbol{G}_{ef} & 0 \\ \boldsymbol{H}_{fe} & -\boldsymbol{G}_{ff} & \boldsymbol{I} \\ 0 & \boldsymbol{E} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{t}_f \\ \boldsymbol{u}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{ee} \\ \boldsymbol{G}_{fe} \\ 0 \end{bmatrix} \left[\boldsymbol{t}_e \right]. \qquad (11)$$

In the final modification the known and unknown boundary conditions are rearranged. The modified matrix equation is solved and the unknown displacements and tractions along the external boundary and the displacements and interaction forces for fibres are obtained.

3. Numerical examples

To demonstrate the accuracy of the method and possible applications three numerical examples are solved.

3.1. Rigid fibre in an infinite plate

An infinite plate with a rigid fibre of length $2l$ is subjected to the parallel loading q_1 or to the perpendicular loading q_2, as shown in Fig. 6. The structure is modelled as a finite square plate with a rigid fibre and the dimensions of the plate are 10 times larger than the fibre. The material of the plate has the Poisson ratio ν=0.25 and is in plane stress state. The rigid fibre is divided into 20 boundary elements and the square plate into 160 boundary elements. Stresses are computed at 245 internal points in the neighbourhood of the fibre. This field is marked as a grey square in Fig. 6. The contour plot of normalized stresses σ_{11}/q_1 for the parallel loading q_1 and normalized stresses σ_{22}/q_2 for the perpendicular loading q_2 are shown in Fig. 7.

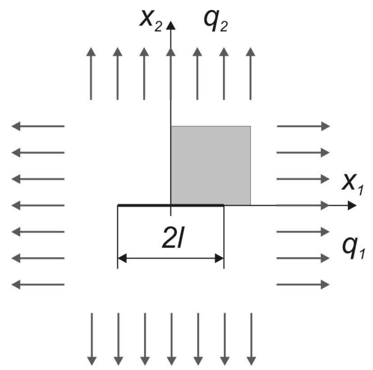


Figure 6: Rigid fibre in an infinite plate – dimensions and loading

a)

b)

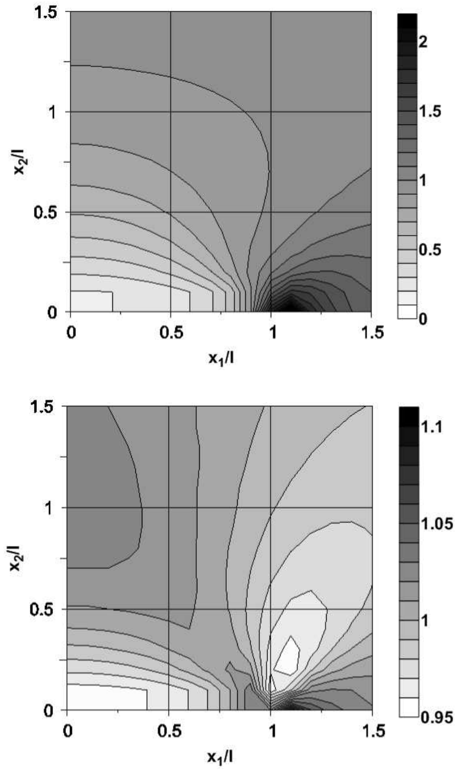


Figure 7: Normal stresses in the vicinity of the rigid fibre:
a) parallel loading – normalized stress σ_{11}/q_1,
b) perpendicular loading – normalized stress σ_{22}/q_2

For the parallel loading stresses σ_{11} have the smallest values along the fibre and the largest values along the extension of the fibre. For the perpendicular loading the stresses σ_{22} are more uniformly distributed in the analyzed field. The maximum values of stresses exceed the applied tension by about 2%. The distribution of stresses agree very well with stresses computed analytically and presented by Pingle et al. [6].

3.2. Two rigid fibres in an infinite plate

Two parallel rigid fibres in an infinite plate are subjected to tension q_1 (Fig. 8). The length of the fibres is $2l$ and the distance between the fibres is $2d_2$. The same material properties and discretization are assumed as in the previous example 3.1. The influence of the distance between the fibres on stress distribution is studied. The normalized stresses σ_{11}/q_1, in the marked field shown in Fig. 8, for the distances d_2/l=0.5 and d_2/l=1.0 are presented in Fig. 9. For the considered distances the interaction of fibres has small influence on stress distribution.

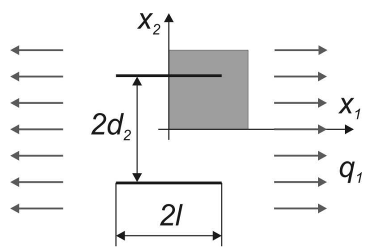


Figure 8. Two rigid fibres in an infinite plate – dimensions and loading

a)

b)

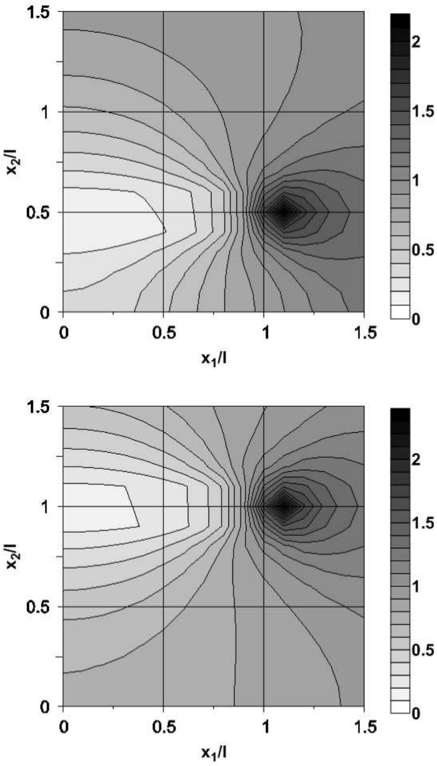


Figure 9. Influence of the distance between two fibres on the stress distribution σ_{11}/q_1: a) d_2/l=0.5, b) d_2/l=1.0

3.3. *Rigid fibres in a rectangular plate*

A rectangular plate of length *2b* and height *2h* contains 13 rigid fibres of length *2l*, as shown in Fig. 10. The horizontal distance between centres of neighbouring fibres is d_1 and the vertical distance is d_2. The ratios of dimensions are: b/l=5, h/l=4, d_1/l=3 and d_2/l=1.6. The material of the plate has the Poisson ratio ν=0.3 and is in plane strain state. The plate is subjected to the horizontal loading q_1. Each rigid fibre is divided into 8 boundary elements and the external boundary into 72 boundary elements. The initial shape and the deformed shape of the plate are shown in Fig. 11. The stress distributions in the matrix, in the marked field in Fig. 10, are shown in Fig. 12. The relative effective Young modulus is computed as E_c/E_m=1.345, where E_c and E_m are the Young modulus of the composite and the matrix, respectively. The effective Poisson ratio of the composite is the same as for the matrix.

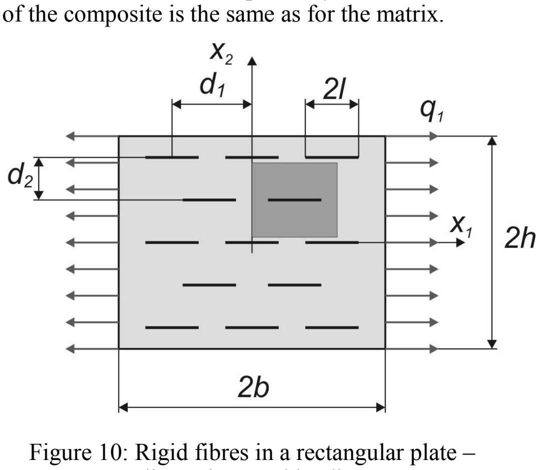


Figure 10: Rigid fibres in a rectangular plate – dimensions and loading

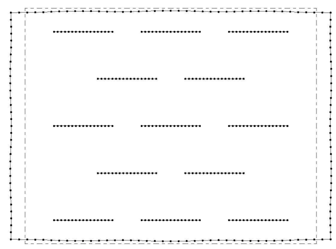


Figure 11. Rigid fibres in a rectangular plate – initial shape (dashed line) and deformed shape (continuous line)

a)

b)

c)

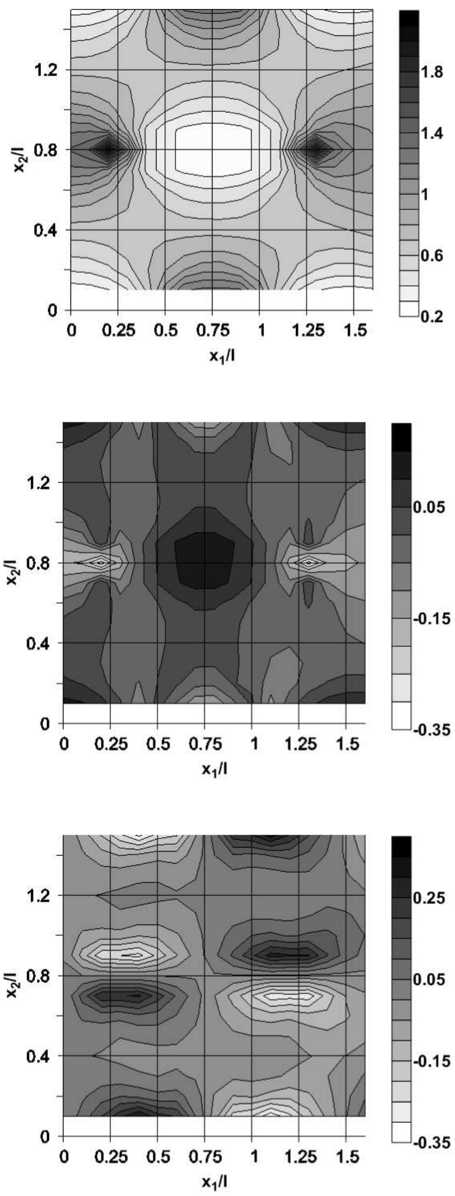
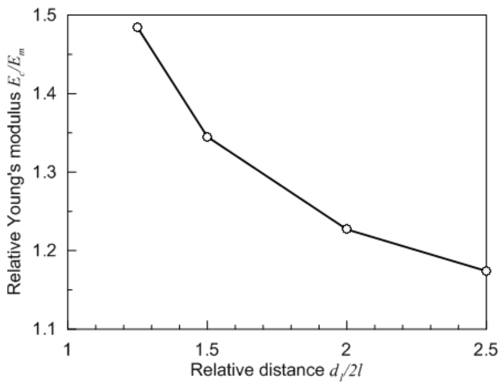


Figure 12. Stress distributions in the matrix: a) σ_{11}/q_1, b) σ_{22}/q_1, c) σ_{12}/q_1

The influence of the horizontal and vertical distance between the fibres on the effective Young modulus is analyzed. The dimensions of fibres are constant and the dimensions of the plate are changed proportionally to the distance between the fibres. The dependence of the relative Young modulus on the horizontal and vertical distance is presented in Fig. 13. If the horizontal or vertical distance are smaller than half of the length of the fibre the stiffness of the composite significantly increases. For the increasing distance between the fibres the stiffness of the composite tends to the stiffness of the matrix.

a)



b)

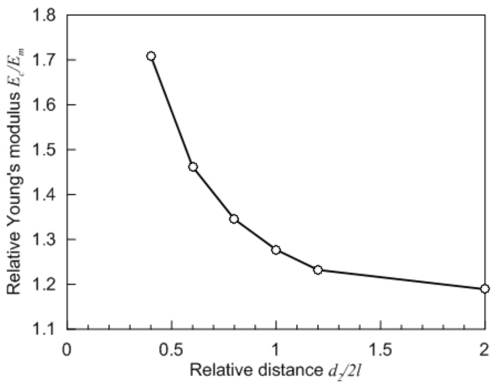


Figure 13. Influence of the distance between the fibres on the relative effective Young modulus E_c/E_m: a) horizontal distance $d_1/2l$, b) vertical distance $d_2/2l$

4. Conclusions

The boundary element method is a very efficient method for modelling elastic composites with rigid inclusions, particularly fibres. The modelling of composites is significantly simplified in comparison to domain methods, for example the finite element method, because nodes are situated only along the external boundary and fibers. Positions, shapes and dimensions of fibers and their number can be very simply modified.

The comparison of stresses for the fibre in the infinite plate obtained by the present method and the analytical method shows that the proposed approach gives very accurate results. For distances between the fibres smaller than the half length of fibres an influence of interaction of fibres on stress distribution in the matrix and significant increase of stiffness can be observed.

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