Topology optimization of trusses using bars exchange method

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Abstract

The algorithm of optimization of trusses is presented in the paper, where for topology optimization bars exchange method is used. In the first case, the problem aimed at cost minimization with constraint set on global stiffness is formulated. In the second case, the problem of minimizing the cost function subject to stress and cross-sectional area constraints is discussed and here the multiple-load case is taken into consideration. The conditions for introduction of topology modification and its acceptance are specified. The paper is illustrated with three examples.

Keywords: trusses, algorithm of optimization, optimal topologies, bars exchange

1. Introduction

The problem of optimal design of trusses is considered in the paper. For topology optimization, bars exchange method is proposed. Here, using topological derivative, additional virtual bars are selected. Next, including existing and virtual bars, statically admissible fields of bar forces are created in order to minimize the objective function. This method is the extension of the approach formulated in [2] and [3].

2. Problem formulation and topology modification condition

2.1. Minimum cost problem with stiffness constraints

The problem of optimal design for trusses requiring minimum of material cost with constraint set on global stiffness, can be written as

$$\min C = \sum_{i=1}^{n} c_i A_i l_i \ , \quad \text{subject to } U = \frac{1}{2}\sum_{i=1}^{n}\frac{N_i^2 l_i}{E_i A_i} \le U_0 \, , \tag{1}$$

where C is the total cost of the structure, A_i, l_i, c_i, E_i are the cross-sectional area, length, specific material cost and Young's modulus of the $i-th$ bar, U denotes strain energy and U_0 is the allowable strain energy.

Using the Lagrangian

$$C^* = C + \lambda(U - U_0)\, , \tag{2}$$

we obtain the optimality conditions, with respect to cross-sectional areas and Lagrange multiplier $\lambda(\lambda \ge 0)$, in the form

$$\frac{\partial C^*}{\partial A_i} = c_i l_i - \lambda \frac{dU}{dA_i} = 0,$$

$$\lambda\Big(\frac{1}{2}\sum_{i=1}^{n}\frac{N_i^2 l_i}{E_i A_i} - U_0\Big) = 0, \;\; \lambda \ge 0\, . \tag{3}$$

Now, we derive formula for sensitivity of the strain energy with respect to cross-section areas i.e. for dU/dA_i . For these purpose, we assume that the i-th bar force in general non-statically determinate truss can be presented in the form

$$N_i = N_i^{(0)} + \sum_{k=1}^{K}\alpha_k N_i^{(k)}, \tag{4}$$

where $N_i^{(0)}$ is the i-th bar force in statically determinate truss (the basic truss) i.e. the analyzed truss with removed redundant bars and K is the number of these bars. Moreover, $\alpha_k(A_j)$ are the forces in redundant bars and $N_i^{(k)}$ in k-th state, which corresponds to the basic truss loaded by self-equilibrated system of two unit forces introduced instead of k-th redundant bar. So, the strain energy can be presented in the form

$$U(\alpha_k, A_i) = \frac{1}{2}\sum_{i=1}^{n}\frac{\left(N_i^{(0)} + \sum_{k=1}^{K}\alpha_k N_i^{(k)}\right)^2 l_i}{E_i A_i} \tag{5}$$

and now, we get that

$$\frac{dU}{dA_i} = \frac{\partial U}{\partial A_i} + \sum_{k=1}^{K}\frac{\partial U}{\partial \alpha_k}\frac{\partial \alpha_k}{\partial A_i}\, . \tag{6}$$

Taking into account, that in view of Castigliano theorem

$$\frac{\partial U}{\partial \alpha_k} = 0, \qquad k = 1,2,...,K\, , \tag{7}$$

we obtain sensitivity formula for arbitrary truss in the form (cf.[1])

$$\frac{dU}{dA_i} = \frac{\partial U}{\partial A_i} = -\frac{N_i^2 l_i}{2E_i A_i^2}\, . \tag{8}$$

So, in view of (8), the optimality conditions (3) can be rewritten as follows

$$\frac{\partial C^*}{\partial A_i} = c_i l_i - \frac{1}{2}\lambda\frac{N_i^2 l_i}{E_i {A_i}^2} = (c_i - \frac{1}{2}\lambda\varepsilon_i^2 E_i) l_i = 0,$$

$$\lambda\Big(\frac{1}{2}\sum_{i=1}^{n}\frac{N_i^2 l_i}{E_i A_i} - U_0\Big) = 0, \;\; \lambda \ge 0\, . \tag{9}$$

Calculating from (9) cross-sectional areas and Lagrange multiplier in function of bar forces, we have

$$A_i = \frac{1}{U_0} \frac{|N_i| \sum_{j=1}^{n} \frac{|N_j| l_j \sqrt{c_j}}{\sqrt{E_j}}}{\sqrt{c_i}\sqrt{E_i}},$$

$$\lambda = \left(\frac{1}{U_0} \sum_{i=1}^{n} \frac{|N_i| l_i \sqrt{c_i}}{\sqrt{2E_i}} \right)^2. \tag{10}$$

Now, the cost function, both for the statically determinate and non-determinate trusses, can be expressed as

$$C = \frac{1}{U_0} \left(\sum_{i=1}^{n} |N_i| l_i \sqrt{\tfrac{c_i}{E_i}} \right)^2. \tag{11}$$

When all bars are made of the same material, so that

$$E = E_1 = E_2 = ... = E_n, \quad c = c_1 = c_2 = ... = c_n \tag{12}$$

the cost function has the form

$$C = \frac{c}{EU_0} \left(\sum_{i=1}^{n} |N_i| l_i \right)^2. \tag{13}$$

Using topological derivative approach, the condition of topology modification by introducing of a new $n+1-th$ bar of Young's modulus E_{n+1} and specific cost c_{n+1} can be expressed as follows

$$\left. \frac{\partial C^*}{\partial A_{n+1}} \right|_{A_{n+1}=0} < 0, \quad \text{or} \quad |\varepsilon_{n+1}| > \sqrt{\frac{2c_{n+1}}{\lambda E_{n+1}}}, \tag{14}$$

in which ε_{n+1} denotes virtual strain value along the line connecting the respective nodes. When all bars are made of the same material, the condition (14) takes the form

$$|\varepsilon_{n+1}| > \varepsilon_b, \text{ where } \varepsilon_b = |\varepsilon_1| = |\varepsilon_2| = ... = |\varepsilon_n|. \tag{15}$$

2.2. Minimum cost problem with stress and cross-sectional area constraints

The problem of optimal design for trusses requiring minimum of material cost with stress and cross-sectional area constraints, is formulated in the form

$$\min\ C = \sum_{i=1}^{n} c_i A_i l_i, \text{ subject to } \left|\sigma_i^{(j)}\right| \le \sigma_{ai}^{(j)},\ A_i \ge A_{i\,\min}, \tag{16}$$

in which the notation is the same as in Section 2.1. Furthermore, $\sigma_i^{(j)} = \frac{N_i^{(j)}}{A_i}$ denote the actual stresses and

$$\sigma_{ai}^{(j)} = \begin{cases} R_{ei} & for \quad \sigma_i^{(j)} \ge 0 \\ \dfrac{\pi^2 E_i}{s_i^2} & for \quad \sigma_i^{(j)} < 0, \quad s_i \ge s_i^{cr} \\ R_{ei} - \frac{1}{2} R_{ei} \left(\frac{s_i}{s_i^{cr}} \right)^2 & for \quad \sigma_i^{(j)} < 0, \quad s_i < s_i^{cr} \end{cases} \tag{17}$$

are the admissible stresses in the $i-th$ bar for the $j-th$ loading case $(j = 1,..., j_0)$. R_{ei} is the conventional yield limit. The last expression of (17) corresponds to the Johnson-Ostenfeld formula for the critical buckling stress in the elasto-plastic range. Moreover,

$$s_i = l_i \sqrt{\frac{A_i}{I_i}}, \ s_i^{cr} = \pi \sqrt{\frac{2E_i}{R_{ei}}} \tag{18}$$

denote the slenderness and critical slenderness values, and I_i is the minimal moment of inertia of the bar cross-section.

We assume the relation between the moment of inertia and the cross-sectional area in the form

$$I = \xi A^m, \quad m = 1,2,3,... \tag{19}$$

where ξ is the constant parameter.

Accounting for (18) and (19) in (17) the admissible stresses take the form

$$\sigma_{ai}^{(j)} = \begin{cases} R_{ei} & for \quad \sigma_i^{(j)} \ge 0 \\ \dfrac{\pi^2 E_i \xi_i A^{m-1}}{l_i^2} & for \quad \sigma_i^{(j)} < 0, \quad s_i \ge s_i^{gr} \\ R_{ei} - \dfrac{R_{ei}^2}{4E_i\pi^2} \dfrac{l_i^2}{\xi_i A_i^{m-1}} & for \quad \sigma_i^{(j)} < 0, \quad s_i < s_i^{gr} \end{cases} \tag{20}$$

and the derivatives of admissible stresses are

$$\frac{\partial \sigma_{ai}^{(j)}}{\partial A_i} = \begin{cases} 0 & for \quad \sigma_i^{(j)} \ge 0 \\ \dfrac{(m-1)\pi^2 E_i \xi_i A_i^{m-2}}{l_i^2} & for \quad \sigma_i^{(j)} < 0, \quad s_i \ge s_i^{gr} \\ \dfrac{(m-1) R_{ei}^2 l_i^2}{4E_i \pi^2 \xi_i A_i^m} & for \quad \sigma_i^{(j)} < 0, \quad s_i < s_i^{gr}. \end{cases} \tag{21}$$

Introducing nonnegative Lagrange multipliers $\mu_i \ge 0$, $\eta_i \ge 0$, the augmented objective function can be expressed as

$$C^*(A_i, \mu_i, \eta_i) = C + \sum_{i=1}^{n} \mu_i \left(\left|\sigma_i^{(j)}\right| - \sigma_{ai}^{(j)} \right) + \sum_{i=1}^{n} \eta_i (A_{\min} - A_i). \tag{22}$$

Now, the optimality conditions, with respect to cross-sectional areas and Lagrange multipliers, take the form

$$\frac{\partial C^*}{\partial A_i} = c_i l_i + \mu_i \left(\frac{\partial \left|\sigma_i^{(j)}\right|}{\partial A_i} - \frac{\partial \sigma_{ai}^{(j)}}{\partial A_i} \right) = 0, \quad i = 1,...,n,$$

$$\mu_i \left(\left|\sigma_i^{(j)}\right| - \sigma_{ai}^{(j)} \right) = 0, \quad i = 1,...,n,$$

$$\eta_i (A_{\min} - A_i) = 0, \quad i = 1,...,n. \tag{23}$$

In order to solve (23) with respect to A_i, μ_i and η_i we can divide bars into four groups, namely:

- group 1 corresponding to tensile bars of cross-sectional areas greater than $A_{\min}$;
- group 2 corresponding to compressive bars of slenderness not less than the critical slenderness s_i^{cr} and of cross-sectional areas greater than $A_{\min}$;
- group 3 corresponding to compressive bars of slenderness less than the critical slenderness s_i^{cr} and of cross-sectional areas greater than $A_{\min}$;
- group 4 of bars of cross-sectional areas $A = A_{\min}$.

Taking that $m = 2$, which corresponds to size variation of similar cross-sections, we get

- for group 1

$$A_i = \frac{N_i^w}{R_{ei}}, \quad \mu_i = \frac{c_i}{R_{ei}^2} N_i^w l_i, \quad \eta_i = 0; \tag{24}$$

- for group 2

$$A_i = \frac{\sqrt{\left|N_i^w\right|} l_i}{\pi\sqrt{E_i \xi_i}}, \quad \mu_i = \frac{c_i l_i^3}{2\pi^2 E_i \xi_i}, \quad \eta_i = 0; \tag{25}$$

- for group 3

$$A_i = \frac{\left|N_i^w\right|}{R_{ei}} + \frac{R_{ei} l_i^2}{4E_i \pi^2 \xi_i}, \quad \mu_i = \frac{c_i l_i}{R_{ei}}\left(\left|N_i^w\right| + \frac{R_{ei}^2 l_i^2}{4E_i \pi^2 \xi_i}\right),$$

$$\eta_i = 0; \tag{26}$$

- for group 4

$$A_i = A_{\min}, \quad \mu_i = 0, \quad \eta_i = c_i l_i; \tag{27}$$

where N_i^w is the normal force in the $i-th$ bar sizing its cross-sectional area corresponding to the worst loading case for this bar.

So, using (24) - (27) the cross-sectional area A_i, in terms of bar forces, can be expressed as follows

$$A_i = \begin{cases} \frac{\left|N_i^w\right|}{R_{ei}} + \frac{R_{ei} l_i^2}{4E_i \pi^2 \xi_i}, & N_i^w \le -\frac{l_i^2 R_{ei}^2}{4E_i \pi^2 \xi_i} \\ \frac{\sqrt{\left|N_i^w\right|} l_i}{\pi\sqrt{E_i \xi_i}}, & -\frac{l_i^2 R_{ei}^2}{4E_i \pi^2 \xi_i} \le N_i^w \le -\frac{\pi^2 \xi_i E_i A_{i\min}^2}{l_i^2} \\ A_{i\min}, & -\frac{\pi^2 \xi_i E_i A_{i\min}^2}{l_i^2} \le N_i^w \le R_{ei} A_{i\min} \\ \frac{1}{R_{ei}} N_i^w, & N_i^w \ge R_{ei} A_{i\min}. \end{cases} \tag{28}$$

Substituting (24) - (27) into the objective function (16), it can be expressed, for the statically determinate trusses, in the following form

$$C = \sum_{(1)} \frac{c_i}{R_{ei}} N_i^w l_i + \sum_{(2)} \frac{c_i}{\pi\sqrt{\xi_i E_i}} \sqrt{\left|N_i^w\right| l_i^2} + \sum_{(3)} \frac{c_i}{R_{ei}}\left[\left|N_i^w\right| l_i + \frac{R_{ei}^2 l_i^3}{4E_i \pi^2 \xi_i}\right] +$$

$$+ \sum_{(4)} c_i A_{i\min} l_i, \tag{29}$$

where $\sum_{(i)}$ denotes the summation with respect to all bars belonging to group i.

The condition of topology modification by introduction of $n+1-th$ bar in tension takes the form (14) or (15). In the case of bar in compression it can be presented as

$$\left|\varepsilon_{n+1}\right| > \varepsilon_{0_{n+1}}, \tag{30}$$

where $\varepsilon_{0_{n+1}} = \frac{\pi^2 \xi_{n+1} A_{(n+1)\min}}{l_{n+1}^2}$ denotes strain appearing in the virtual bar in compression of the minimal cross-sectional area.

3. The algorithm of optimal truss design

The algorithm presented here corresponds to one loading case $(j_0 = 1)$, however it can be easily generalized to many loading cases. The following steps can be distinguished in the solution algorithm.

1. Structure analysis

Determination of bar forces can be executed using for example FEM. The analysis problem for a truss is specified by the equilibrium equation

$$\mathbf{Ku=P}, \tag{31}$$

where **K** is the stiffness matrix**, u** is the nodal displacement vector, and **P** is the load vector.

2. Redistribution of cross-sectional areas

The optimal cross-sectional areas A_i for the problem (1) or (16) are calculated respectively from $(10)_1$ or (28).

3. Choice of virtual bars

The virtual connections, where we can introduce new bars are determined using modification conditions (14), (15) or (30).

4. Topology optimization

When a new bar is introduced, in order to avoid increase of bars number, another bar should be deleted. It is executed through the procedure of bar exchange. Instead of each $I-th$ virtual bar self-equilibrated force system $\alpha\hat{N}_I$ is introduced, where $\hat{N}_I$ is the unit force and α is the load factor. The unit force system induces in truss bars the normal forces $\alpha\hat{N}_i$, which are next calculated.

The procedure of topology optimization is conducted by the iterative scheme consisting of the following steps:

A. Calculation of the total bar forces

The total bar forces are calculated from the following formula

$$N_i^c = N_i^w + \alpha\,\hat{N}_i. \tag{32}$$

Using conditions $N_i^c = 0$ we get values of parameter α corresponding to the disappearance of normal forces in consecutive bars.

B. Determination of the objective function variation

For each virtual bar and each value of parameter α the objective function variation is determinated

$$\Delta C = C^{(opt)} - C^{(i)} < 0 \tag{33}$$

where $C^{(opt)}$ denotes the value of C corresponding to the new topology and $C^{(i)}$ is the value of C at the previous step before topology variation. Under the condition $\Delta C < 0$, the change (changes) corresponding to the biggest decrease of the ΔC is chosen.

C. Determination of bar forces after modification

For each bar exchange, that generate the biggest decrease of the objective function value, bar forces are determinated analogously as in step 4A.

D. Termination of the topology optimization procedure

If any bars exchange is not possible, i.e. $\Delta C = 0$ the next step is conducted, otherwise we go to step 4A.

5. Configuration optimization

For given truss topology the optimization of truss configuration is performed.

6. Termination of the algorithm

If any modification of topology cannot be introduced the optimization procedure is terminated, otherwise we go to step 1.

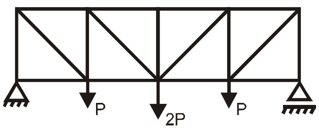
4. Examples

4.1. Example 1 – Optimization of topology and configuration of truss with respect to cost minimization under stiffness constraint

Consider the cost optimization problem of the form described by (1) for the truss presented in Fig. 1a. Three lower nodes of the truss were loaded by vertical forces $P, 2P, P$, where $P = 10^3 N$. The truss consists of 17 bars and it is made of linearly elastic material of Young's modulus value $E = 2.1 \cdot 10^{11} N/m^2$. The length of all horizontal and vertical bars is $l = 1m$, while the length of inclined bars equals $\sqrt{2}l$.

The topology optimization by bars exchanges gives 80 equivalent designs presented in Fig. 1b. For each design the objective function reaches the same value. The removal of zero bars system from optimal topologies, will decrease their number to 20. After topology modification, the cost reduction is about 12%. Finally, the optimal design (Fig. 1c) is obtained by configuration optimization. Now, the cost is about 33% smaller than that of the initial design.

a)



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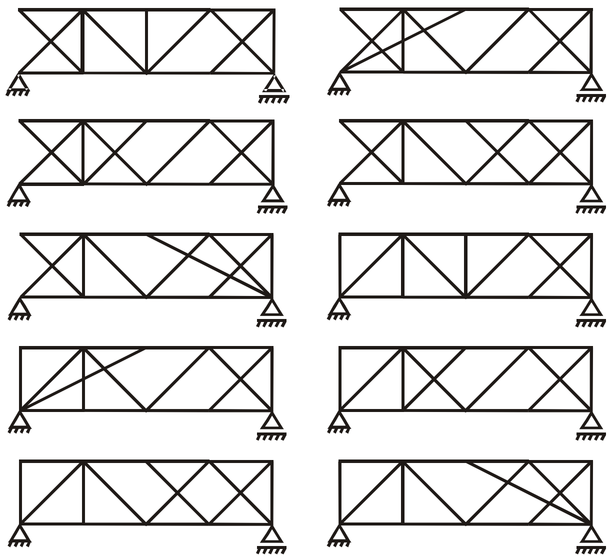

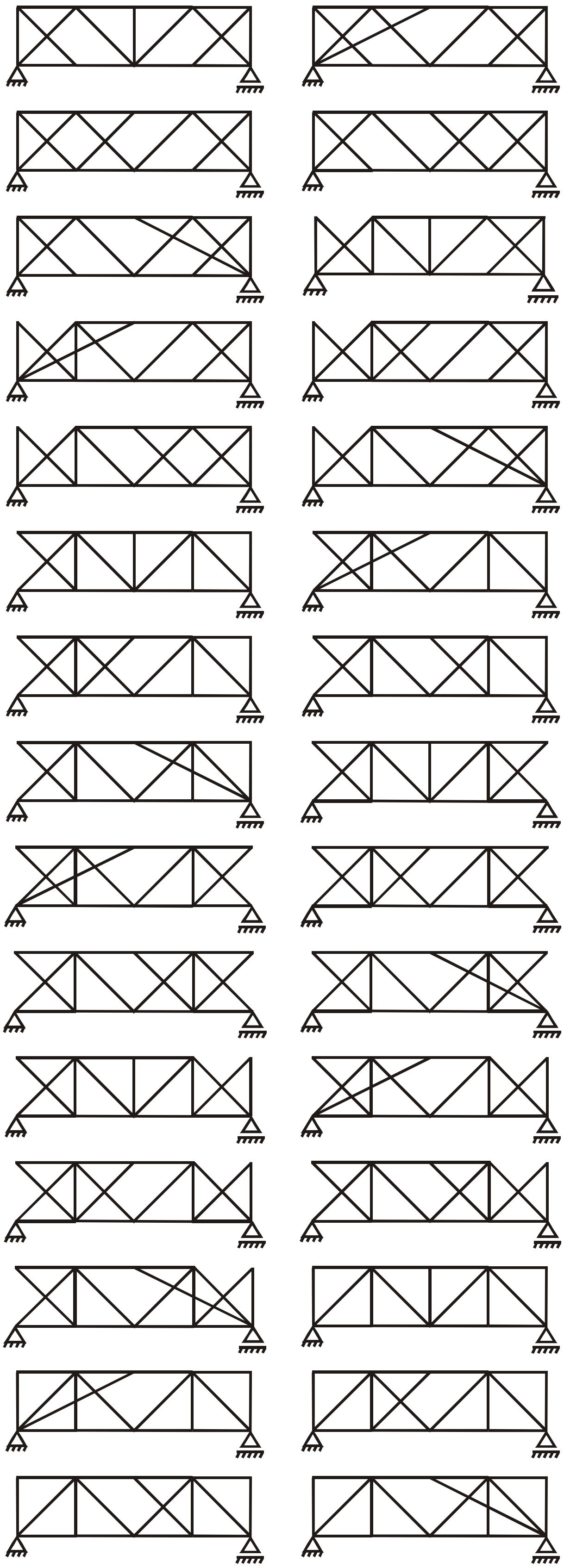

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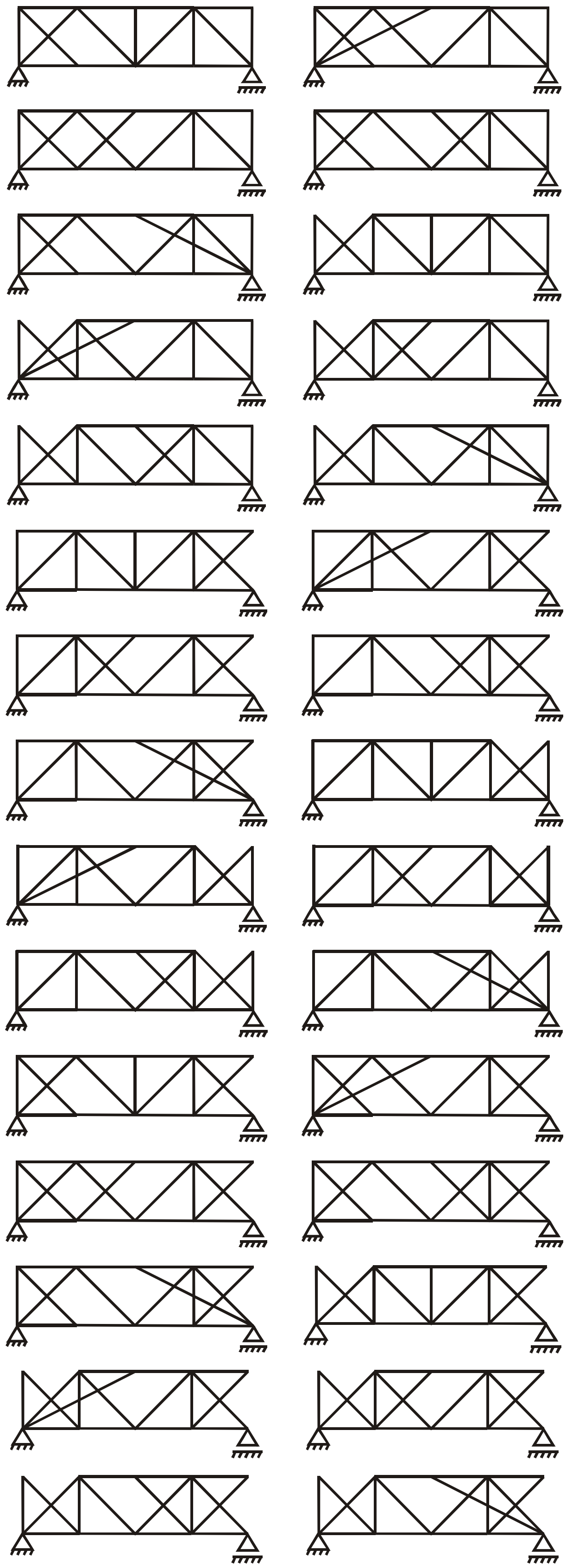

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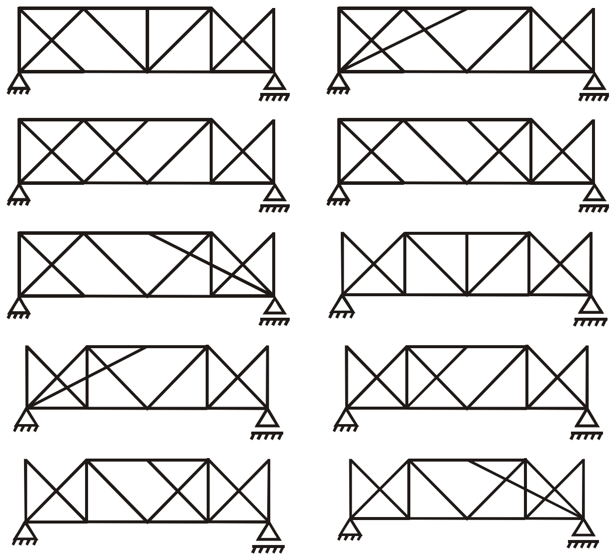

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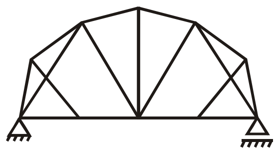


Figure 1: Optimization of the truss: a) The initial design; b) Optimal topologies; c) The optimal design

4.2. Example 2 – Optimization of topology and configuration of truss with respect to cost minimization under stress and buckling constraints for one loading case

Consider the cost optimization problem of the form described by (16) for the 21-bars truss presented in Fig. 2a. Here, one loading case is analyzed ($j_0 = 1$), namely five upper nodes of the truss are loaded by vertical forces P and five lower nodes by vertical forces $5P$, where $P = 10^3 N$. The length of all horizontal and vertical bars is $l = 0.5m$, while the length of inclined bars equals $\sqrt{2}l$. The Young's modulus value is the same as in the previous example. The yield limit is $R_e = 2.5 \cdot 10^8 N/m^2$. The minimal cross-sectional area is $A_{\min} = 0.2 \cdot 10^{-4} m^2$. We assume that all cross-sections have circular shape and in this case, the coefficient of the relation between the moment of inertia and the cross-sectional area is $\xi = 1/(4\pi)$.

The topology optimization by bars exchanges leads to the structure presented in Fig. 2b. Further cost reduction can be achieved by configuration optimization. The final optimal design is shown in Fig. 2c. Now, the cost is 30% smaller than that of the initial design, while topology optimization gives a cost decrease about 23%.

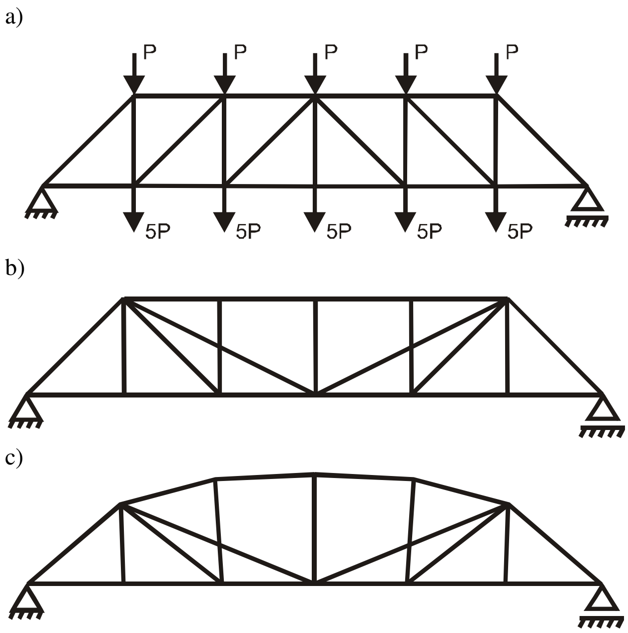


Figure 2: Optimization of the truss: a) The initial design; b) Optimal topology; c) The optimal design

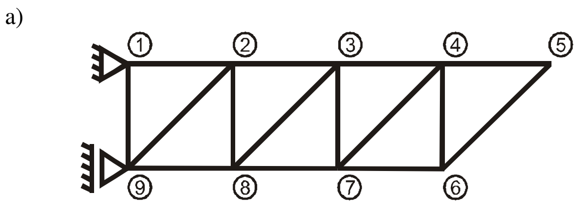
4.3. Example 3 – Optimization of topology and configuration of truss with respect to cost minimization under stress and buckling constraints for two loading cases

Let us consider now the cost optimization problem of the form described by (16), for the 15-bars truss (Fig. 3a) and two loading cases ($j_0 = 2$). Details concerning applied loads are given in Table 1. The material parameters and the minimal cross-sectional area values are the same as in the previous example.

Table 1: Truss loading conditions, forces are given in [N]

Load case	Node number	P_x	P_y
1	2	0	-40 000
	3	0	-30 000
	4	0	-20 000
	5	0	-10 000
2	5	0	-50 000

At first, the optimization procedure is carried out for each load case applied independently. Next, the multi-load case is taken into account and procedure is performed simultaneously for both load cases.



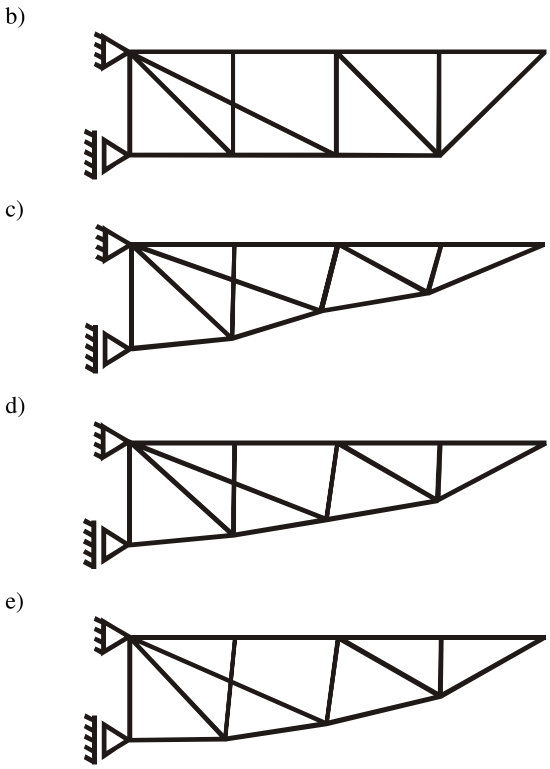


Figure 3: Optimization of the truss: a) The initial design; b) Optimal topology; c) The optimal design for the first loading case; d) The optimal design for the second loading case; e) The optimal design for both loading cases

The topology optimization conducted by bars exchange method leads to the structure shown in Fig. 3b. The structure is the same for each considered loading cases. The configuration optimization provides the final optimal designs shown in Fig. 3c, d and e. The percentage decrease of the cost value after topology and configuration optimization with relation to the initial value of the cost is presented in Table 2.

Table 2: The cost value decrease, given in [%]

Load case	Topology optimization	Configuration optimization
1	19.8	30.3
2	25.7	29.7
1 and 2	19.8	25.4

5. Conclusions

The heuristic algorithm of optimization of trusses is presented and tested in this paper. Here, topology optimization is performed using bars exchange method. It provides a new and effective tool of optimal design of trusses. The characteristic feature of this approach is possibility of simultaneous generation of many equivalent optimal topologies, which especially appear for trusses containing repeated bar subsystems. Moreover, bar forces for a new topology are calculated using previous solution, so this method ensures significant reduction of the computational time.

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