A posteriori error estimator of finite element method for advection-diffusion-reaction problems: piecewise linear approximations on triangles

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Abstract

We consider the construction of piecewise defined a posteriori error estimators (AEE) for finite element approximations of advection-diffusion-reaction boundary value problems. The key feature is the ability of the Galerkin scheme to do the approximation error estimation analysis for the arbitrarily selected triangulation element independently. This method of error estimation is illustrated for two dimensional case when AEE of piecewise linear approximation is constructed in the form of the second degree polynomial.

Keywords: boundary value problems, error estimation, finite element methods, Galerkin methods

1. Introduction

A posteriori error estimates are an intellectual part of the h -adaptive finite element methods (FEM) [1-4]. In this article we construct AEE for piecewise linear approximations of advection-diffusion-reaction boundary value problems.

2. Boundary value problem and its variational formulation

Let Ω be a connected bounded domain of the Euclidean space R^2 with Lipchitz continuous boundary $\Gamma = \partial\Omega$. We will consider the following boundary value problem for the advection-diffusion-reaction equation:

$$\begin{cases} \text{Find } u = u(x) \text{ such that} \\ -\nabla.[\mu\nabla u] + \beta.\nabla u + \sigma u = f \text{ in } \Omega \subset R^2, \\ \qquad\qquad u = 0 \text{ on } \Gamma \equiv \partial\Omega. \end{cases} \tag{1}$$

The variational formulation of the problem (1) reads: find the function $u \in V$ such that

$$a_\Omega(u,v) = \langle l_\Omega, v\rangle \qquad \forall v \in V , \tag{2}$$

where

$$\begin{cases} V := H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \right\}, \\ a_\Omega(w,v) := \int_\Omega [\mu\nabla w.\nabla v + v\beta.\nabla w + \sigma w v]\,dx, \\ \langle l_\Omega, v\rangle := \int_\Omega f v\,dx \qquad \forall v,w \in V. \end{cases} \tag{3}$$

Here $\mu = \{\mu_{ij}(x)\}_{i,j=1}^2, \beta = \{\beta_i(x)\}_{i=1}^2, \sigma = \sigma(x)$ and $f = f(x)$ are given functions such that almost everywhere in Ω :

$$\mu_{ij} = \mu_{ji}, \;\; \nabla.\beta := \sum_{i=1}^2 \partial\beta_i / \partial x_i = 0, \;\; \sigma \ge 0, \tag{4}$$

$$\sum_{i,j=1}^2 \mu_{ij}\xi_i\xi_j \ge \mu_0 \sum_{i=1}^2 \xi_i^2 \;\; \mu_0 = \text{const} > 0 \;\; \forall \xi_i \in R, \tag{5}$$

$$\mu_{ij}, \beta_i, \sigma \in L^\infty(\Omega), \quad f \in L^2(\Omega). \tag{6}$$

If we take into account (4)-(6) it is possible to prove that the bilinear form $a_\Omega(.,.): V \times V \to R$ generates the norm

$$\|v\|_V := \sqrt{a_\Omega(v,v)} \quad \forall v \in V . \tag{7}$$

3. Discrete problem

We assume that $\Omega \subset R^d$ is partitioned into finite elements so that $\Omega = \bigcup_{K\in\Im_h} K$, $\Im_h = \{K\}$, $h := \max_{K\in\Im_h} \operatorname{diam} K$ and

(i) $K \cap K' = \varnothing \quad \forall K, K' \in \Im_h : K \ne K'$;

(ii) $\bar{K} \cap \bar{K}' = \begin{cases} S := \{\text{shared edge of } K \text{ and } K'\} \\ A := \{\text{shared vertex of } K \text{ and } K'\} \;\; \forall K, K' \in \Im_h . \\ \varnothing \end{cases}$

Having constructed the finite element space $V_h \subset V$, $\dim V_h = N(h) = N < +\infty$ corresponding to the triangulation $\Im_h = \{K\}$ we seek an approximation $u_h \in V_h$ for the solution $u \in V$ of the problem (2) in the following way:

$$\begin{cases} \text{suppose we have } \Im_h = \{K\} \text{ and } V_h \subset V; \\ \text{find } u_h \in V_h \text{ such that } a_\Omega(u_h, v) = \langle l_\Omega, v\rangle \;\; \forall v \in V_h. \end{cases} \tag{8}$$

4. A posteriori error estimation problem

In view of the variational problem (2) and its discrete problem (8) it is easy to formulate such a variational problem [1]: find approximation error $e := u - u_h \in E := V \setminus V_h$ such that

$$\begin{cases} a_\Omega(e,v) = \langle \rho(u_h), v\rangle \quad \forall v \in E, \\ \langle \rho(w), v\rangle := \langle l_\Omega, v\rangle - a_\Omega(w,v) \quad \forall w,v \in V. \end{cases} \tag{9}$$

In order to solve the problem (9) we apply the Galerkin procedure by using some finite dimensional subspace within E :

$$\begin{cases} \text{find error estimator } e_h \in E_h \subset E := V \setminus V_h \text{ such that} \\ a_\Omega(e_h, v) = \langle \rho(u_h), v\rangle \quad \forall v \in E_h. \end{cases} \tag{10}$$

Problem (10) is well-posed and its solution $e_h \in E_h$ can be characterized by property

$$\|e_h\|_V \le \|u - u_h\|_V \qquad \forall h > 0. \tag{11}$$

5. AEE refined structure on triangle

5.1. *Piecewise linear approximations on triangle*

Let us assume that the domain $\Omega \subset R^d$ is partitioned into finite elements K so that the resulting triangulation $\Im_h = \{K\}$ has properties (i)-(ii) and let us denote by $A_h = \{A_i\}_{i=1}^n$ set of all vertices $A_i := (x_1^i, x_2^i)$. We also assume that the approximation $u_h \in V_h := \{v \in C(\Omega) : v|_K \in P_1(K)\ \forall K \in \Im_h\}$ of the solution (2) is obtained and u_h is of the following form:

$$u_h(x)|_K = \sum_{m=i,j,k} u_m L_m(x) \quad \forall x = (x_1, x_2) \in K \quad \forall K \in \Im_h, \tag{12}$$

where $P_1(K)$ is the space of polynomials of total degree less or equal to 1 on the triangle $K = A_i A_j A_k$,

$$\begin{aligned} &L_i(x) := \frac{a_i + b_i x_1 + c_i x_2}{2|K|}, \\ &|K| := \frac{1}{2}(b_1 c_2 - b_2 c_1), \\ &a_i := x_1^k x_2^j - x_1^j x_2^k, \\ &b_i := x_2^j - x_2^k, \\ &c_i := x_1^k - x_1^j, \quad i \to j \to k \to i. \end{aligned} \tag{13}$$

5.2. *A posteriori error estimator on triangle*

Assuming that the finite element approximation $u_h \in V_h$ has sufficiently precise values in triangulation nodes A_i we construct a posteriori error estimator e_h of the error $e(x) = u(x) - u_h(x)$ of the following form:

$$e_h(x) := \sum_{K \in \Im_h} e_K(x) = \sum_{K \in \Im_h} \lambda_K \phi_K(x) \quad \forall x \in \Omega, \tag{14}$$

where

$$\begin{cases} \operatorname{supp} \phi_K := K, \\ \phi_K(x) := 3\left[L_i(x)L_j(x) + L_j(x)L_k(x) + L_k(x)L_i(x)\right], \\ \forall x \in K \quad \forall K \in \Im_h. \end{cases} \tag{15}$$

Coefficients λ_K can be calculated by sequential reviewing of the triangulation $\Im_h = \{K\}$ elements and by following the next rule:

$$\begin{aligned} \lambda_K &\equiv e_h(x_K) = \frac{\langle l_K, \phi_K\rangle - a_K(u_h, \phi_K)}{\|\phi_K\|_V^2} \\ &\cong 2|K| \left.\frac{6|K| f - \sum_m u_m\left[3(\beta_1 b_m + \beta_2 c_m) + 2|K|\sigma\right]}{\mu \sum_m \left(b_m^2 + c_m^2\right) + \frac{16}{5}|K|\sigma}\right|_{x = x^K}, \end{aligned} \tag{16}$$

where x^K is a mass center of the triangle K.

6. Numerical results

Let us consider the model problem with homogeneous Dirichlet boundary condition and input parameters given in Table 1.

Table 1: Problem input parameters

μ	$\beta = (\beta_1, \beta_2)$	σ	$f(x,y)$	Ω
$\{\delta_{ij}\}_{i,j=1}^2$	$(0,0)$	-10	$100xy$	$(0,1)^2$

Here δ_{ij} is Kronecker delta.

We compute approximations on the sequence of uniformly refined triangular meshes and after that we find convergence rates of approximations following the next rule [3]:

$$P\left[H^m(\Omega), e_{h/2}\right] := \log_2 \frac{\| e_h \|_{m,\Omega} - \| e_{h/2} \|_{m,\Omega}}{\| e_{h/2} \|_{m,\Omega} - \| e_{h/4} \|_{m,\Omega}}, \; m = 0, 1. \tag{17}$$

Table 2: Convergence of approximations calculated on the sequence of uniformly refined triangular meshes, here $N := card\, \Im_h$, $\tilde{u}_h = u_h + e_h$, $p_1 := P[V, e_h]$, $p_2 := P[V, u_h]$, $p_3 := P[V, \tilde{u}_h]$

N	$\| e_h \|_V$	$\| u_h \|_V$	$\| \tilde{u}_h \|_V$	p_1	p_2	p_3
8^2	3.975	9.109	9.938	-	-	-
16^2	1.945	9.560	9.756	-	-	-
32^2	0.969	9.686	9.735	1.1	1.8	3.1
64^2	0.484	9.720	9.732	1.0	1.9	2.7
128^2	0.242	9.728	9.731	1.0	2.0	2.4
256^2	0.121	9.730	9.731	1.0	2.0	2.3

For example, from the above Table 2 it is clear that AEE approximation norm has first convergence rate in the space V – what we needed to prove. Adding a posteriori error estimator e_h to the approximated solution u_h gives us better convergence rates on coarse triangular meshes.

The suggested AEE established a reputation as a reliable and effective tool for finding h-adaptive FEM approximations to advection-diffusion-reaction boundary value problems. We prove this fact on the basis of results of the numerical experiments.

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