Plasticity of Soil in Finite Elements Modeling

Sergei Klovanich[1] **and Natalia Jankowska**[1]

[1]*Department of Mechanics and Fundamentals of Building Design,*
Faculty of Technical Sciences, University of Warmia and Mazury in Olsztyn
ul. Prawocheńskiego 19, 10-720 Olsztyn, Poland
e-mail:sergei.klovanich@uwm.edu.pl

Abstract

The phenomenological model for a soil in the form of the associated theory of the plasticity, based on a loading surface of the closed form, is formulated. The analytical form of this surface is offered. Dilatancy are considered, deformation hardening and softening. The paper is focused on nonlinear analysis using finite elements method. The examples of calculations, confirming reliability of model, are proposed.

Keywords: constitutive model of soil, plasticity, finite elements method

1. Introduction

The classical theories of plasticity, established on hypotheses of the Mohr-Coulomb, Drukker-Prager, Mizes-Botkin, apply to the description of strength surfaces linear forming. As a result strength of material at uniform hydrostatic compression is infinite. Thus the plastic flow associates with a surface loading, which results with modification of a strength surface and too is opened at hydrostatic compression. However, from many researches follows, that the loading surface is closed both at isotropic tension, and at hydrostatic compression.
Here generalization is given of a well known Drukker-Prager's model symmetric of diagonal of space of the main stresses, having forming in the form of the closed curve and section in the form of a curvilinear triangle.

The theory of a plastic flow with hardening is formulated in the form of relations between increments of deformations $d\{\varepsilon\}$ and stresses $d\{\sigma\}$ and defined as [1]

$$d(\sigma)=[D]_{ep}d\{\varepsilon\} \tag{1}$$

where $[D]_{ep}$ - elastic-plastic stiffness matrix

$$[D]_{ep}=[D]-\frac{[D]\dfrac{\partial Q}{\partial\{\sigma\}}\left(\dfrac{\partial F}{\partial\{\sigma\}}\right)^T[D]}{\left(\dfrac{\partial F}{\partial\{\sigma\}}\right)^T[D]\dfrac{\partial Q}{\partial\{\sigma\}}+\Lambda} \tag{2}$$

$[D]$ – initial elastic matrix, corresponding to the Hook's law for an isotropic material; F and Q - loading function and plastic potential (for associated theory $Q=F$); Λ - hardening function. If a hardening measure χ is the work of stresses on the plastic deformations $\{\varepsilon_p\}$, and $d\chi=\{\sigma\}^T d\{\varepsilon_p\}$, that hardening function is defined so

$$\Lambda=\frac{\partial F}{\partial\chi}\{\sigma\}^T\frac{\partial F}{\partial\{\sigma\}}=\frac{\partial F}{\partial\{\varepsilon_p\}}\frac{\partial F}{\partial\{\sigma\}} \tag{3}$$

The plastic strain rate is normal to the surface, represented by loading function F. Let's consider, so-called, associated version of flow theory, when $Q=F$. In this case F represents a surface, on a normal to which there is a plastic flow. In case to take into consideration of hardening this surface always passes through a point current stresses and changes during loading. The surface is formulated in local cylindrical octahedrical coordinates system σ_o,τ_o,θ, where θ - so-called, angle of a type of the stresses condition $0\le\theta\le\pi/3$

$$\sigma_o=\frac{1}{3}(\sigma_x+\sigma_y+\sigma_z);\quad \tau_o=\sqrt{\frac{2}{3}}\sqrt{J_2};$$
$$\theta=\frac{1}{3}arccos\left(\sqrt{2}\,\frac{J_3}{\tau_o^3}\right) \tag{4}$$

Here J_2 and J_3 - the second and the third invariantes of stresses deviator

$$J_2=-s_xs_y-s_ys_z-s_xs_z+\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2;$$
$$s_x=\sigma_x-\sigma_o;\ s_y=\sigma_y-\sigma_o;\ s_z=\sigma_z-\sigma_o;$$
$$J_3=s_xs_ys_z-s_x\tau_{yz}^2-s_y\tau_{xz}^2-s_z\tau_{xy}^2-2\tau_{xy}\tau_{yz}\tau_{xz}.$$

In this system of coordinates $\{\sigma\}=\{\sigma_o\ \tau_o\}$ и $\{\varepsilon\}=\{\varepsilon_o\ \gamma_o\}$, where

$$\varepsilon_o=\frac{1}{3}(\varepsilon_x+\varepsilon_y+\varepsilon_z);\gamma_o=2\sqrt{\frac{2}{3}}\sqrt{J_{2\varepsilon}};$$
$$J_2=-e_xe_y-e_ye_z-e_xe_z+\frac{1}{4}\left(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{xz}^2\right);$$
$$e_x=\varepsilon_x-\varepsilon_o;\ e_y=\varepsilon_y-\varepsilon_o;\ e_z=\varepsilon_z-\varepsilon_o,$$

ε_o и γ_o - octahedrical longitudinal and shear deformations; $J_{2\varepsilon}$ - the second invariant of deformations divator. The initial matrix of elasticity, entering into expression (2), in this case will look like

$$[D]=\begin{bmatrix} 3K_o & 0 \\ 0 & G_o \end{bmatrix} \tag{5}$$

where K_o and G_o - the volumetric modulus and the shear modulus.

Having carried out matrix multiplication and addition, expression (2) it is possible to present in the usual form

$$[D]_{ep}=\begin{bmatrix} 3K & 0 \\ 0 & G \end{bmatrix} \tag{6}$$

where

$$3K=3K_o-\lambda\left(a_{11}+a_{12}\frac{d\varepsilon_o}{d\gamma_o}\right)/\Delta; G=G_o-\lambda\left(a_{22}+a_{12}\frac{d\gamma_o}{d\varepsilon_o}\right)/\Delta.$$

Here

$$a_{11}=9K_o^2\left(\frac{\partial F}{\partial\sigma_o}\right)^2; a_{12}=3K_oG_o\frac{\partial F}{\partial\sigma_o}\frac{\partial F}{\partial\tau_o}; a_{22}=G_o^2\left(\frac{\partial F}{\partial\tau_o}\right)^2$$

$$\Delta=-\frac{a_{11}+a_{12}}{3K_o}-\frac{\partial F}{\partial\chi}\left(\sigma_o\frac{\partial F}{\partial\sigma_o}+\tau_o\frac{\partial F}{\partial\tau_o}\right).$$

2. Loading function

Expression for a loading surface is accepted as follows

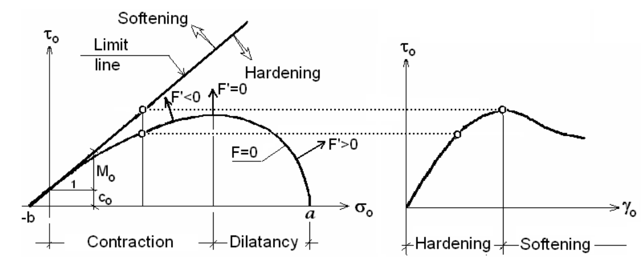

Figure1: a)-Loading function, b) – diagram $\tau_o-\gamma_o$

$$F(\sigma_o\tau_o\theta)=\tau_o-A_o(b+\sigma_o)\sqrt{a-\sigma_o}=0 \tag{7}$$

This surface always passes through a point of current loading σ_o,τ_o, closed from two sides and crosses an axis σ_o in points –*b* and *a* (fig. 1). In addition on function (4) following conditions are imposed

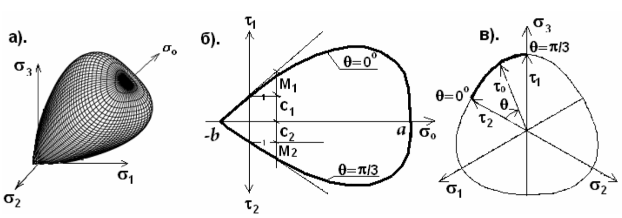

Figure 2: General view of a surface and its intersections

$$\tau_o\big|_{\sigma_o=0}=c_o;\quad \partial\tau_o/\partial\sigma_o\big|_{\sigma_o=0}=M_o, \tag{8}$$

which allow to establish following relations between parameters of a equation (7)

$$A_o=\frac{2\sqrt{a}}{2a+b}M_o;\qquad b=\frac{2ac_o}{2aM_o-c_o}. \tag{9}$$

The parameter a also can be established from (7). The curve (7) asymptotive comes nearer to a limiting failure line on Mohr $\tau_o=M_o\sigma_o+c_o$ (Fig.1). Thus the site under a this line corresponds to a hardening zone, above a straight line – softening zone. On the stresses-strains diagram $\tau_o-\gamma_o$ (Fig. 1b) it conforms to an ascending and descending branches. Besides, the derivative of function (4) changes a sign on a site from $-b$ to a. Change of a sign of a derivative is the transfer from contraction to dilatancy. General view of loading surface and its characteristic sections are presented on fig. 2. So-called, deviational section of this surface is a curvilinear triangle with maksimal τ_1 and minimal τ_2 radiuses (fig. 2), corresponding to extreme values of a corner $\theta=\pi/3$ at $\sigma_1=\sigma_2>\sigma_3$ and $\theta=0$ at $\sigma_1>\sigma_2=\sigma_3$.
Thus on the Coulomb

$$M_1=2\sqrt{2}\frac{sin\varphi}{3-sin\varphi};\; M_2=2\sqrt{2}\frac{sin\varphi}{3+sin\varphi};$$
$$c_1=2\sqrt{2}\frac{cos\varphi}{3-sin\varphi}c;\; c_2=2\sqrt{2}\frac{cos\varphi}{3+sin\varphi}c,$$

where φ and c - corner of an internal friction and coupling.
Between two limiting cases $\theta=0$ and $\theta=\pi/3$ it is possible to present interpolation so [4]

$$\rho(\theta)=1-4(1-g)cos\beta\;(1-cos\beta),\quad \beta=\frac{\pi}{3}-\theta \tag{10}$$

where $g=\dfrac{3-\sin\varphi}{3+\sin\varphi}$. Thus $\rho(0)=1$ and $\rho(\pi/3)=g$. Then parameters of the formula (7) can be defined so $M_o=\rho(\theta)M_1,\quad c_o=\rho(\theta)c_1$,
and

$$A_o=\rho(\theta)A_1;\quad A_1=\frac{2\sqrt{a}}{2a+b}M_1.$$

Parameter a can be established exactly from (8), but it can be set according by simpler dependence

$$a\approx\frac{\sigma_o}{1-[\tau_o/(M\sigma_o)]^2}+b \tag{11}$$

3. Derivatives of loading function

The vector-column of derivatives $\frac{\partial F}{\partial\{\sigma\}}$, entering into expressions (2) and (3), looks like

$$\frac{\partial F}{\partial\{\sigma\}} = \left\{ \frac{\partial F}{\partial \sigma_o} \quad \frac{\partial F}{\partial \tau_o} \right\} \tag{12}$$

Differentiating (7) on σ_o and τ_o, we shall have

$$\begin{aligned} &\frac{\partial F}{\partial \sigma_o} = -A_o\left(\sqrt{a-\sigma_o} - \frac{b+\sigma_o}{2\sqrt{a-\sigma_o}}\right); \\ &\frac{\partial F}{\partial \tau_o} = 1-(b+\sigma_o)\sqrt{a-\sigma_o}\,\frac{\partial A_o}{\partial \tau_o}. \end{aligned} \tag{13}$$

$$\frac{\partial A_O}{\partial \tau_O} = A_1\frac{\partial \rho(\theta)}{\partial \tau_O} = A_1\frac{\partial \rho(\theta)}{\partial \theta}\frac{\partial \theta}{\partial \tau_O} = \frac{3\sqrt{2}\,J_3\,A_1}{\tau_O^4\sqrt{1-2\left(\frac{J_3}{\tau_O^3}\right)^2}}\frac{\partial \rho(\theta)}{\partial \theta}$$

Differentiation of expression (10) gives

$$\frac{\partial \rho(\theta)}{\partial \theta} = -4(1-g)\sin\vartheta(2\cos\vartheta - 1) \tag{14}$$

Thus, all analytical expressions for the derivatives entering into equations parities (2) and (3), are established.

4. Hardening and softening

Change of porosity in process of loadings and unloadings is taken as additional measures of hardening. Thus volume porosity of a ground is defined depending on octahedrical normal pressure [3,5] $e = e^o - \mu\ \ln\sigma_o$ - at active loading; $e = e^A - \delta\ln\sigma_o$ - at unloading, where e^o - porosity in natural state; e^A - porosity of a ground by the time of unloading; μ и δ - the parameters, defined from tests. The volumetric strains, caused by change of porosity, are defined so

$$3\varepsilon_o = -\frac{e-e_o}{1+e}, \qquad 3d\varepsilon_o = -\frac{de}{1+e}. \tag{15}$$

Increment of a plastic part of the volumetric strains connected with change of porosity, are equal $d\varepsilon_o^p = d\varepsilon_o - d\varepsilon_o^e$, where $d\varepsilon_o^e$ - elastic strains. In other words,

$$3d\varepsilon_o^p = \frac{\mu-\delta}{(1+e)\sigma_o}d\sigma_o \tag{16}$$

From (16) it is possible to receive expression for a derivative, entering into a equation (3)

$$\frac{\partial F}{\partial \chi} = \frac{3(1+\mathrm{e})}{\mu-\delta}\frac{\partial F}{\partial \sigma_O} \tag{17}$$

At softening the derivative $\frac{\partial F}{\partial \chi}$ is defined from the stresses-strains diagrams (fig.1b) $\frac{\partial F}{\partial \chi} = \frac{1}{\tau_O}\frac{\partial F}{\partial \tau_O}\frac{\partial \tau_O}{\partial \gamma_O}$.

Let's notice, that unloading is carried out under the nonlinear, logarithmic law with modulus $K_O = \frac{1+e}{\delta}\sigma_O;\ G_O = \frac{3K_O(1-2\nu)}{2(1+\nu)}$, where ν - Poisson's ratio. Thus the second composed in (2) is equal to zero.

If stresses lay above a strength line (pic.1a), i.e. are in a zone of softening lose significance since thus known postulate Drukkera is broken. Process of deformations, nevertheless, proceeds and described by means of a falling branch of the diagram (fig.1в). In this case expression (17) we shall present in the form of

$$\frac{\partial F}{\partial \chi} = \frac{1}{\tau_o}\frac{\partial F}{\partial \tau_o}\frac{\partial \tau_o}{\partial \gamma_o^p} = \frac{1}{\tau_o}G_p\frac{\partial F}{\partial \tau_o} = \frac{1}{\tau_o}\frac{G_oG}{G_o-G}\frac{\partial F}{\partial \tau_o} \tag{18}$$

where G_p - the plastic shear modulus. Tangent plastic shear modulus G shall define, having set analytical expression for a curve on fig.1в, in the form of, similar for the diagram of deformation softening materials at uniaxial compression [6]

$$\xi = \frac{k\eta}{1+(k-2)\eta+\eta^2} \tag{19}$$

where $\xi = \frac{\tau_o}{\bar{\tau}_o}$; $\eta = \frac{\gamma_o}{\bar{\gamma}_o}$; $k = G_o\frac{\bar{\gamma}_o}{\bar{\tau}_o}$. Here $\bar{\gamma}_o$ and $\bar{\tau}_o$ - coordinates of top of the diagram. Differentiating (19) on γ_o, shall receive

$$G = \frac{\partial \tau_o}{\partial \gamma_o} = G_o\left[\frac{1-\eta^2}{\left[1+(k-2)\eta+\eta^2\right]^2}\right] \tag{20}$$

5. Model testing

The model is included in computer program "Concord" [4], realizing a finite elements method in nonlinear statement.

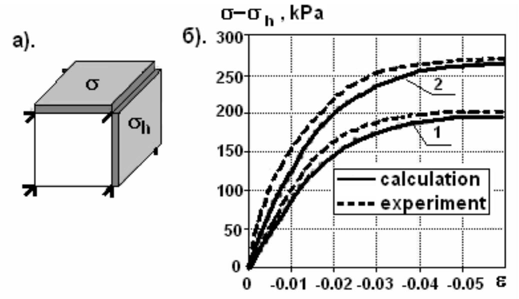

Figure 3: The first test

As the first test example calculation of volumetric finite element of the rectangular form of the (fig.3a) . Loads were put to an element in horizontal and vertical directions by steps, and at the first step hydrostatic compression σ_h =100 kPa was created. On the following steps the vertical load down to before destruction. On fig.3b data of calculations for a soil with following characteristics are presented: 1. E=20 MPa, ν =0.2, $\varphi=30^o$, c=0; 2. E=30 MPa, ν =0.2, $\varphi=35^o$, c=0. Results of calculations were compared to data [7].

In the second example the problem about a stamp acting on a ground was considered [5]. The sizes of a stamp are 40x40 sm. Steps of loading are 40 kN/m^2. For calculation volumetrics isoparametrical eight-nodes finite elements were used. The symmetric part of the sample was considered only. Ground characteristics: E=2.5 MPa, φ =18^o, c=0.045 MPa, ν =0.35. Results of calculation of vertical displacements of a stamp are presented on fig.5. Here isolines of displacements (fig. 5a) and stresses (fig. 5b,c) are given at q=680 kN/m^2.

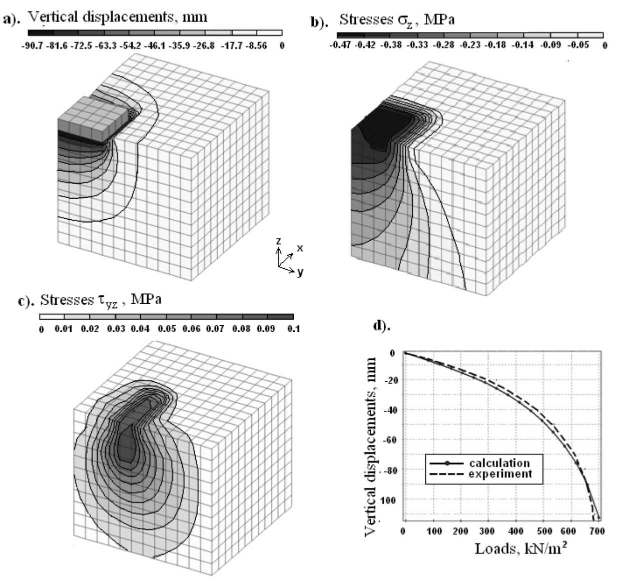

Figure 5: The second test

References

[1] Zienkiewicz O.C. and Taylor, R.L., *The Finite Element Method*, Vol. 2 *Solid Mechanics*, fifth ed., – Butterworth-Heinmann, Oxford, 2000.

[2] Terzaghi, K., and Ralph B. Peck, 1948, *Soil Mechanics in Engineering Practice*, second ed., - John Wiley and Sons, New York, 1967.

[3] Roscoe K.J. and Schofield A.N. Machanical Behavior of an Idealised "Wet" Clay, *Proc. European Conf. on Soil Mechanics and Foundation Engineering, v.1, pp.47-54, 1963*.

[4] Klovanich S.F. *Finite elements method in nonlinear problems of engineering mechanics*, - Zaporozhie (in Russian), 2009.

[5] Balura M.V. Horizontal dispacements of the basis under a rigid stamp, *Osnovania, fundamenty i mehanika gruntov,* №6, *pp.39-41,(in Russian) 1973*.

[6] Liu T.C.Y., Nilson A.H., Slate F.O. Stress-Strein Response and Fracture of Concrete in Uniaxial and Biaxial Compression//ACI Journal. – v.69.- 1972.- pp. 291-295.

[7] Burland J.B. The Yielding and Dilatation of Clay.(Correspondence)// Geotechnique.- 1965. – V.15. – P.211-214.