Model of hyperelastic fibre-reinforced material

Stanisław Jemioło[1] **and Marcin Gajewski**[2]

[1]*Department of Strength of Materials, Theory of Elasticity and Plasticity, Warsaw University of Technology*
al. Armii Ludowej 16, 00-637 Warsaw, Poland
e-mail: s.jemiolo@il.pw.edu.pl

[2]*Department of Strength of Materials, Theory of Elasticity and Plasticity, Warsaw University of Technology*
al. Armii Ludowej 16, 00-637 Warsaw, Poland
e-mail: m.gajewski@il.pw.edu.pl

Abstract

A class of relatively simple constitutive models of hyperelastic non-homogeneous composite materials (isotropic matrix reinforced with continuous fiber families) is proposed. Analysed class of constitutive models in approximation reduces to the classical one for linear theory model of fibrous composites, in which full bounding between matrix and fiber is assumed as well as one-dimensional fiber deformation. Specific model is proposed in the paper altogether with some details of model implementation in FEM system ABAQUS. The strain energy potential for this model is a poly-convex function, what ensure existence of solutions of boundary value problems for hyperelasticity and good numerical conditioning. Constitutive relationships expressed in objective incremental form are implemented in FORTRAN in user procedure UMAT. The numerical tests are proposed to check correctness of the implementation with available analytic solutions, and some boundary value problems are also solved.

Keywords: hyperelasticity, anisotropy, fibre reinforced composite, computational mechanics, finite element method

1. Introduction

Composite materials obtained by reinforcing of isotropic matrix by fibers are commonly used in technical applications. The matrix function is fulfilled by synthetic resins characterized with considerable elongation (e.g. epoxide resins, polyester resins) while fibers are made of glass, carbon, boron and organic materials. The main goal of insertion of fiber into matrix is exceeding of materials stiffness and strength. The fundamental condition to obtain planed mechanical properties of composite is a good coupling between components (in this case between fibers and resin). And this feature was a starting point a dozen or so years ago for proposition of fiber composite theoretical model based on mixture theory, cf. e.g. [2,4]

The fundamental motivation for development of anisotropic hyperelastic constitutive models was and still is biomechanics and mechanics of woven materials. Application of large deformation theory is needed especially for geosynthetics, where assumption about lack of distinction between actual and reference configuration correct for small deformation theory is not still valid. In case of biomechanics and soft tissue description application of large deformation theory is well-founded, cf. [5] and literature cited there. Unfortunately, in other mechanic disciplines need for large deformation theory application is not always well understood. What is incomprehensible, particularly in frame of finite element method and computer development.

The simplest theory in which all theoretical requirements are fulfilled is theory of hyperelastic materials, and a class of constitutive models for fiber reinforced materials considered herein is situated in group of anisotropic hyperelastic constitutive models (especially orthotropic and transversally isotropic hyperelastic materials, c.f. [3,4,5], where application of representation of anisotropic tensor function results is in place). The main goal of this paper is to extend previously presented by authors constitutive model (c.f. [4]) in such a way to allow analysis of composites in which matrix is reinforced with many fiber families.

2. The fundamental assumptions and constitutive model

According to basic assumption, the elastic strain energy function (ES) of hyperelastic material reinforced with several fiber families can be postulated as follows:

$$\Psi = \left(1-\sum_{n=1}^{N} p_n\right)\Psi_M + \sum_{n=1}^{N} p_n \Psi_{Zn} , \tag{1}$$

where Ψ_M is an elastic strain function (isotropic one) for matrix (ESM) and Ψ_{Zn} are energy strain functions of fiber families (ESZ) while p_n stands for volume ratio of fibers in material volume unit. We assume that $N_{\max}=6$, and that matrix material is an isotropic material, for which ESM function is an isotropic one with respect to right deformation tensor $\mathbf{C}=\mathbf{F}^T\mathbf{F}$, and left deformation tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^T$, where $\mathbf{F}$ is so called deformation gradient tensor. The tensor $\mathbf{F}$ have a positive determinant $J=\det\mathbf{F}>0$, and symbol "T" in above relations stand for tensor transposition. According to above assumption (isotropic function) the ESM is a function of three non-reducing invariants of deformation tensor:

$$\Psi_M = W\left(I_1, I_2, J\right) , \tag{2}$$

where

$$I_1 = \mathrm{tr}\mathbf{C}=\mathrm{tr}\mathbf{B} \ , I_2 = \mathrm{tr}\left(\mathrm{cof}\mathbf{C}\right)=\mathrm{tr}\left(\mathrm{cof}\mathbf{B}\right) ,\ J=\det\mathbf{F} . \tag{3}$$

The "tr" symbol stands for tensor trace, while "cof" is a transposition of adjoined tensor. The ESZ function of n-th fiber family, which "works" in one direction described with vector $\mathbf{m}_n\left(\mathbf{X}\right)$ is approximated as:

$$\Psi_{Zn} = \frac{E_{Zn}}{4}\left(I_{4n}-1\right)^2 , \tag{4}$$

where $I_{4n} = \mathrm{tr}\hat{\mathbf{M}}_n$, and $\hat{\mathbf{M}}_n$ tensors represent parametric tensors $\mathbf{M}_n = \mathbf{m}_n \otimes \mathbf{m}_n$ in actual configuration, evaluated according to the relation:

$$\hat{\mathbf{M}}_n = \mathbf{F}\mathbf{M}_n\mathbf{F}^T \,. \tag{5}$$

In (4) the E_{zn} parameters have an interpretation of Young modulus of n-th fiber family. In this paper we are considering a special case of ESM function, so called Ciarlet model for compressible materials:

$$\Psi_M = \frac{\mu_o}{2}\left[f\left(I_1-3\right)+\left(1-f\right)\left(I_2-3\right)\right]+\frac{1}{4}\left[\lambda_o-2\mu_o\left(1-f\right)\right]J^2+ \\ -\left[\frac{1}{2}\lambda_o+\mu_o\right]\ln J-\frac{1}{4}\left[\lambda_o-2\mu_o\left(1-f\right)\right], \tag{6}$$

also discussed in [5]. If $f=1$, then from (6) we get Ogden model. The parameters: μ_o and λ_o have interpretation of Lame constants (identical like in small deformation theory). Function (6) is a poly-convex one and fulfil appropriate conditions of elasticity potential growth if and only if, iff $\mu_o > 0$, $f \in (0,1)$ and $\lambda_o > 2\mu_o(1-f)$, cf. [5], and then function (1) based on (4) and (6) is poly-covex. From local energy and mass balance rules (altogether with balance of linear and angular momentum rules) one can obtain the Kirchhoff stress tensor for matrix made of Ciarlet model in the following form:

$$\boldsymbol{\tau}_M = \mu_0 f\,\mathbf{B} + \mu_0\left(1-f\right)\left(I_1\mathbf{B}-\mathbf{B}^2\right) \\ +\left[\frac{1}{2}\left(\lambda_0-2\mu_0\left(1-f\right)\right)J^2-\frac{1}{2}\lambda_0-\mu_0\right]\mathbf{I} \,. \tag{7}$$

Proceeding analogically, it is easy to show that from (4-5) one can obtain the following relation for Kirchhoff stress in n-th fiber family:

$$\boldsymbol{\tau}_{Zn} = E_{Zn}\left(I_{4n}-1\right)\hat{\mathbf{M}}_n \,. \tag{8}$$

So, the constitutive relationship for material reinforced with n-th fiber families (in current configuration) is as follows:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \left(1-\sum_{n=1}^{N}p_n\right)\boldsymbol{\tau}_M + \sum_{n=1}^{N}p_n\boldsymbol{\tau}_{Zn} \,, \tag{9}$$

where $\boldsymbol{\sigma}$ is a Cauchy's stress tensor.

3. Implementation in ABAQUS FEM software

Constitutive model proposed in section 2 is implemented in ABAQUS user procedure UMAT. So constitutive relationship (9) has to be rearranged into incremental form, cf. [4] and ABAQUS manual [1]. After evaluation of objective Jaumann derivative of Kirchhoff stress tensor, one can obtain forth order tensor (stiffness tensor in incremental constitutive relationship).

The programmed UMAT procedure firstly was tested in boundary-value problems with homogeneous stress and strain fields, where analytical solution is possible to obtain. These tests have shown good agreement of obtained numerical results with analytical solutions.

4. Numerical tests -pipe compression

The task of compression of pipe with circumferential cross-section area in the direction of its axis by applying displacement boundary conditions at the bottom and top base is considered. On the other parts of pipe (outside surfaces) the zero stress boundary conditions are assumed. The pipe is made of hyperelastic material with fiber reinforcement, characterized by constitutive relationship expressed by (9). The three cases are considered, i.e. lack of reinforcement, the reinforcement overlaps with direction of the pipe axis and reinforcement is placed circumferentially. It is worth emphasizing that fiber placement directions are given in reference configuration, and during deformation undergo local changes according to (5). In analyzed boundary-value problem the following material data were assumed: $p = 0.05$, μ_0 =1.0 E_M, λ_0 =1.5 E_M, E_Z =26 E_M, where E_M is an initial Young's modulus of matrix material, and f =0.2.

All tasks were solved using standard Newton-Raphson incremental algorithm. The example contour graph of Mises stresses for pipe with circumferential reinforcement in the final pipe is shown in fig.1.

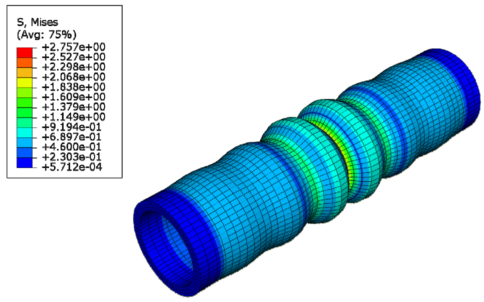

Figure 1: Contour graphs of Mises stresses for pipe with circumferential reinforcement in the final configuration (e.g. u_3 =20 [mm] for 100[mm] pipe)

5. Final remarks

The model of fibre reinforced materials proposed in this paper has been implemented in general FEM software ABAQUS/Standard. Polyconvexity of the stored energy function ensure existence of solutions of hyperelastic BVP.

References

[1] ABAQUS *Theory manual*, Version 6.1., Hibbitt, Karlsson and Sorensen, Inc., Pawtucket, 2000.

[2] Boehler J.P. (ed): *Applications of tensor functions in solid mechanics. CISM Courses and Lectures*, no. 292, Wien-New York, Springer-Verlag, 1987.

[3] Bonet J., Burton A.J.: A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations, *Comput. Methods Appl. Mech. Engrg.*, 162, pp. 151-164, 1998.

[4] Gajewski M., Jemioło S.: FEM implementation of hyperelastic constitutive models of reinforced materials (in Polis), *XVI Slovak-Polish-Russian Seminar Theoretical Foundation of Civil Engineering*, Žilina, str. 119-126, Moscow, 2007.

[5] Jemioło S.: *A study of the hyperelastic properties of isotropic materials. Modelling and numerical implementation (in Polish)*, Prace Naukowe, Budownictwo z. 140, str. 1-308, OWPW, 2002.