Topology optimization with uncertain loading conditions

János Lógó[1], Dániel B. Merczel[2] and Anna Vásárhelyi[3]*

[1]*Department of Structural Mechanics, Budapest University of Technology and Economics*
Műegyetem rkp. 3, H-1111 Budapest, Hungary
e-mail: logo@ep-mech.me.bme.hu

[2]*Department of Structural Mechanics, Budapest University of Technology and Economics*
Műegyetem rkp. 3, H-1111 Budapest, Hungary
e-mail: dmerczel@mail.bme.hu

[3]*Department of Structural Engineering, University of Pécs*
Boszorkány út 2, H-7624 Pécs, Hungary
e-mail: anna.vasarhelyi@freemail.hu

Abstract

A new type of probabilistic optimal topology design method is elaborated where the points of application of the loads are given randomly. In the proposed probabilistic topology optimization method the minimum penalized weight design of the structure is subjected to compliance constraint and side constraints. By the use of an appropriate approximation, the original problem is substituted by a deterministic one. The numerical procedure is based on iterative formula, which is formed by the use of the first order optimality condition of the Lagrangian function. The application is illustrated by numerical examples.

Keywords: topology optimization, optimization, stochastic phenomena, structural mechanics

1. Introduction

The topology optimization has more than 100 years history and still an expanding field in optimal design (Sokół et. al [7]). The majority of the papers deal with deterministic problems. Lógó [1, 3] and Lógó et.al [2, 4] elaborated rather powerful methods for stochastic topology optimization where the magnitude of the loads or the compliance limit are given by their mean values, covariances and distribution functions. Applying an appropriate approximation for the loading uncertainties the stochastic expressions are substituted by equivalent ones. This work is a continuation of the above cited papers.

Here the loading positions are taken as stochastic variables and all the other data are deterministic. To make the optimization method robust an equivalent deterministic problem is derived where the support forces are calculated as stochastic expressions (uncertainty of the magnitudes) and used as loadings while the original loading positions are considered at the point of the mean values as supports.

The resulted design problem is based on the formerly composed loading and support conditions. The iterative solution technique is derived by the optimality criteria method. Several numerical examples are presented and compared.

2. Approximation of a Probabilistic Expression

From the literature the following theory is known (Prekopa [5], Lógó [3]). If $\xi_1, \xi_2, ..., \xi_n$ have a joint normal distribution, then the set of $\mathbf{x} \in \Re^n$ vectors satisfying

$$\mathrm{P}(x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n \leq 0\ ) \geq q\ ; \tag{1}$$

is the same as those satisfying

$$\sum_{i=1}^{n} x_i\mu_i + \Phi^{-1}(q)\sqrt{\mathbf{x}^T\mathbf{K}_{ov}\mathbf{x}} \leq 0 \tag{2}$$

where $\mu_i = E(\xi_i)$, $(i = 1, 2, ..., n)$ is the mean value of the randomly given element ξ_i, $\mathbf{K}_{ov}$ is the covariance matrix of the random vector $\xi^T = (\xi_1, \xi_2, ..., \xi_n)$, q is a fixed probability and $0 \langle q \langle 1$, $\Phi^{-1}(q)$ is the inverse cumulative distribution function (so called probit function) of the normal distribution. Expression (2) is convex, the proof can be found in Prekopa [5].

In the following the above theory is applied.

3. Compliance Design

The deterministic compliance design procedure of a linearly elastic 2D structure (disk) in plane stress is known from literature (e.g. (Rozvany[6,] Lógó [1]). This topology optimization problem is given as follow:

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{3}$$

subject to

$$\begin{cases} \mathbf{u^T F} - C \leq 0; \\ -t_g + t_{\min} \leq 0; & (for\, g = 1, ..., G), \\ t_g - t_{\max} \leq 0; & (for\, g = 1, ..., G). \end{cases}$$

Here the ground element thicknesses t_g are the design variables with lower bound $t_{\min}$ and upper bound $t_{\max}$, respectively. By the use of the FEM discretization, each ground element (g= *1,…,G*) contains several sub-elements *(e=1,…,* E_s), whose stiffness coefficients are linear homogeneous functions of the ground element thickness t_g. Furthermore γ_g is the specific weight and A_g the area of the ground element g. $\mathbf{u^T}$ is the nodal displacement vector associated with the loading **F**, **u**

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can be calculated from $\mathbf{Ku} = \mathbf{F}$, where $\mathbf{K}$ is the system stiffness matrix. p is the penalty parameter ($p \geq 1$) and the given compliance value is denoted by C. The above constrained mathematical programming problem can be solved by the use of SIMP algorithm (Lógó[1]).

4. Probabilistic Compliance Design in case of Uncertain Magnitude of the load

Suppose that the loads are given by n point loads and the loading $\mathbf{F}^{\mathrm{T}} = [f_1, f_2, ..., f_n]$ has uncertainties such that the magnitudes of the elements of $\mathbf{F}^{\mathrm{T}}$ are random and they follow joint normal distribution. The mean values of the elements of $\mathbf{F}^{\mathrm{T}}$ and the elements of the covariance matrix $\mathbf{K}_{ov}$ are denoted by $\bar{f}_i = E(f_i)$; $(i = 1,...,n)$ and $\kappa_{i,j}$; $(i = 1,...,n;\ j = 1,...,n;)$, respectively. The nodal displacement vector $\bar{\mathbf{u}}^{\mathrm{T}} = [\bar{u}_1, \bar{u}_2, ..., \bar{u}_n]$ associated with the loading $\bar{\mathbf{F}}^T = (\bar{f}_1, \bar{f}_2, ..., \bar{f}_n)$ is calculated from $\mathbf{K}\bar{\mathbf{u}} = \bar{\mathbf{F}}$. According to the principle of the virtual forces the displacement u_i can be determined by the use of a unit virtual force acting at the location and direction of f_i. The applied formulation is $u_i = \mathbf{u}^T \mathbf{K} \tilde{\mathbf{u}}_i$ where the displacement $\tilde{\mathbf{u}}_i$ is associated to this unit virtual force.

As it is known, the compliance value can be calculated as follows:

$$\mathbf{u}^T \mathbf{F} = u_1 f_1 + u_2 f_2 + ... + u_n f_n \,, \tag{4}$$

where the displacements (u_i, $i = 1,...,n$) are obtained from $\mathbf{Ku} = \mathbf{F}$ linear system and denote the displacements under the force f_i, $(i = 1,...,n)$ in the direction of this load.

By the use of appropriate approximations and simplifications (Lógó [3]) a possible “first order” stochastic compliance value can be obtained:

$$\mathbf{u}^T \mathbf{F} \sim 2\bar{\mathbf{u}}^T \mathbf{F} - \bar{\mathbf{u}}^T \bar{\mathbf{F}}, \tag{5}$$

This “linearized” expression can be written in extended form as follows

$$\left(2\left(\bar{u}_1 f_1 + ... + \bar{u}_n f_n\right) - \left(\bar{u}_1 \bar{f}_1 + ... + \bar{u}_n \bar{f}_n\right)\right) - C \leq 0 \,, \tag{6}$$

where the first part contains the stochastic expression and the second one is a deterministic form. Using a simple modification this compliance limit equation (eq.6) has to be modified according to the given probability value in the stochastic case. The new constraint can be expressed as follow:

$$\mathrm{P}\left(\left(2\left(\bar{u}_1 f_1 + ... + \bar{u}_n f_n\right) - \left(\bar{u}_1 \bar{f}_1 + ... + \bar{u}_n \bar{f}_n\right)\right) - C \leq 0\right) \geq q \;; \tag{7}$$

where $0 \langle q \langle 1$ is the given probability value.

Then the penalized minimum weight problem subjected to probabilistic compliance constraint has the form:

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{8}$$

$$\text{subject to } \begin{cases} \bar{\mathbf{u}}^T \mathbf{K} \bar{\mathbf{u}} - C + 2\Phi^{-1}(q)\sqrt{\mathbf{x}^T \mathbf{K}_{ov} \mathbf{x}} \leq 0; \\ -t_g + t_{\min} \leq 0; \quad (for\ g = 1,...,G), \\ t_g - t_{\max} \leq 0; \quad (for\ g = 1,...,G). \end{cases}$$

The above constrained mathematical programming problem can be solved by the use of a modified SIMP algorithm (Lógó[3]).

Since the mathematical nature of the problem (8) is similar to a classical topology optimization problem (Lógó[1]) all the mathematical statements concerning convexity and differentiability are valid too (Rozvany [6], Lógó [1,3]). The penalization of the ground element thicknesses t_g results in a more distinct material distribution indicating material or no material. Due to this penalization the optimization problem is non-unique in some sense, but the method is widely applied in engineering optimization.

5. Probabilistic Compliance Design in the Case of Uncertain Loading Positions: Adjoint Design

The method what was described in the previous sections can be extended for the case of uncertain load positions. Here a simple technique is introduced. The elaborated technique can be

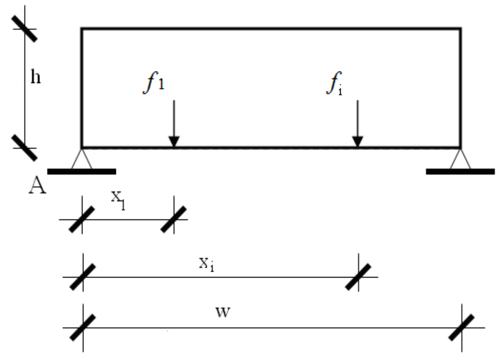

Figure 1. The design domain with the boundary conditions

applied if the structure is supported by a few (two, three) supports. Actually most of the engineering structures have similar boundary conditions.

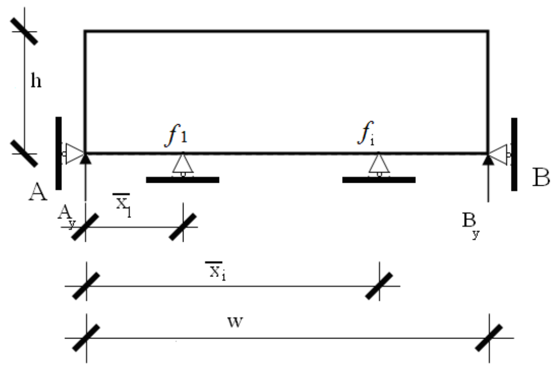

Figure 2. Adjoint structure with the boundary conditions

Let supposed that the structure (the design domain) is supported by two hinges (A and B) and n ($i=1,..,n$) vertical forces act as external loads ($f_1, ..., f_i, ..., f_n$) on it. The distance of the load f_i indicated by x_i as point of application (Figure 1) follows Gaussian distribution. Because the precise value of x_i is not known, x_i is given by its mean value $\bar{x}_i$ and standard deviation σ_i. Due to the stochastic nature of the point of application of the loads the topology optimization can not be elaborated easily.

Instead of solving the problem described above one can create an equivalent (adjoint) structural design problem (see Figure 2) where the vertical forces are substituted by vertical supports at the mean value of the point of application of the corresponding load f_i and the vertical components of the supports substituted by the calculated reaction force components with stochastic magnitudes $\left(A_{y,}\ B_y\right)$. Their mean values:

$$\bar{A}_y = \sum_{i=1}^{n} f_i - \sum_{i=1}^{n} \frac{\bar{x}_i}{w} f_i, \quad \bar{B}_y = \sum_{i=1}^{n} \frac{\bar{x}_i}{w} f_i \tag{9}$$

and standard deviations

$$\bar{\sigma}_A = \bar{\sigma}_B = \sqrt{\sum_{i=1}^{n} \left(\frac{f_i}{w}\right)^2 \sigma_i^2} \tag{10}$$

can be calculated. By the use of these values the design problem turns back to the problem (8) where the magnitude of the loads is probabilistic. For the sake of simplicity the covariance matrix $\mathbf{K}_{ov}$ has only diagonal elements that are different from zero.

6. Numerical examples: Rectangular design domain with two simple supports and two point loads located at the bottom.

The optimization problem is a simple structure given by a rectangular domain (Figure 3.) with two simple supports and two point loads. The double arrows of the loads represent their probabilistic behaviour in Figure 3. They follow Gaussian distribution.

The height/length ratio is 0.35 and the supports are located at the bottom of the left and right corners, respectively. The magnitudes of the loading $\mathbf{F}^T = \left[f_1, f_2\right]$ are varied but they are deterministic values (in symmetric case they are 100 units each

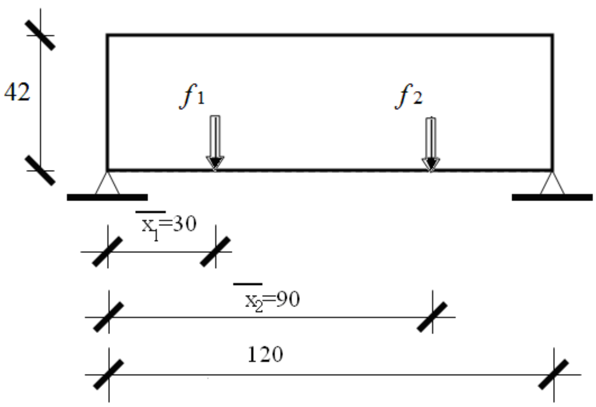

Figure 3. The design domain

and in unsymmetric case the forces are $f_1 = 100$ and $f_2 = 50$ units). The expected locations are at a quarter way along the bottom line. The variations of the distance values can be seen in Table 1 and in Table 2. The rectangular ground structure is dimensionless 120x42 units. The Poisson's ratio is 0. For the sake of simplicity the value of the Young's modulus is assumed to be unity.

In the following the two cases are investigated, where at first the deterministic problem is presented while secondly the stochastic optimal topologies are calculated with different covariant values. The analytical solution can be seen in Figure 4. in symmetric case.

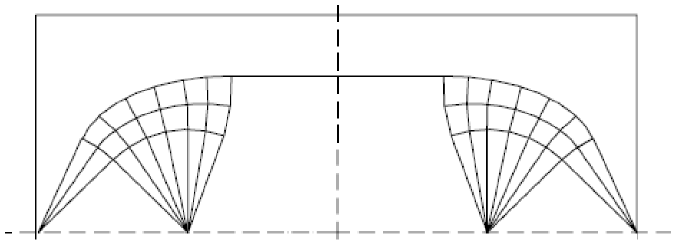

Figure 4. Analytical solution of the deterministic problem in symmetric case

6.1. *Deterministic topology optimization*

120x42 ground elements with 2x2 sub-elements are used. Total number of the finite elements is 20160. The penalty parameter p was run from p=1 to p=2 with smooth increasing (increment is 0.1) and later to p=3 with increment=0.25.

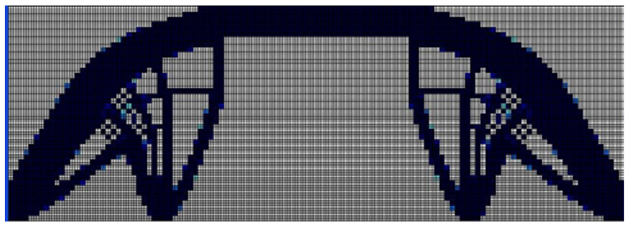

Figure 5. Numerically obtained optimal topology of the deterministic problem in symmetric case

The compliance limit is C=450000. The numerically obtained optimal topology can be seen in Figure 5 in symmetric case while the unsymmetric case is presented in Figure 6. Here the compliance limit is C=300000.

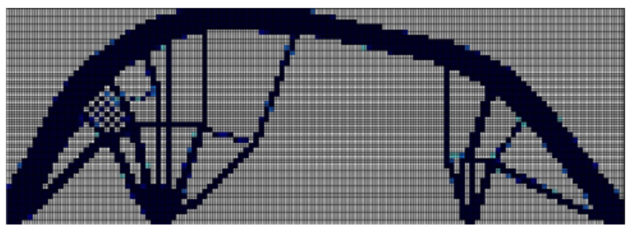

Figure 6. Numerically obtained optimal topology of the deterministic problem in unsymmetric case

6.2. *Probabilistic Topologies with Adjoint Design*

As it was indicated earlier in that case of stochastic topology optimization the magnitudes of adjoint loads are random variables and the point of application is at the original position of the supports. They follow a normal distribution with the calculated mean values (M(A), M(B). Due to the assumed nature of the loading the elements of covariance matrix have different main diagonal values (D(A), D(B)).

Table 1: Data of the symmetric case

				Original	**structure**				**adjoint**	**structure**		
M(x_1)	**M(x_2)**	**%1**	**%2**	**D(x_1)**	**D(x_2)**	**F1**	**F2**	**w**	**M(A)**	**D(A)**	**M(B)**	**D(B)**
30	90	0	0	0	0	100	100	120	100	0	100	0
30	90	5	5	1,5	1,5	100	100	120	100	1,767767	100	1,767767
30	90	10	10	3	3	100	100	120	100	3,535534	100	3,535534

Table 2: Data of the unsymmetric case

				Original	**structure**				**adjoint**	**structure**		
M(x_1)	**M(x_2)**	**%1**	**%2**	**D(x_1)**	**D(x_2)**	**F1**	**F2**	**w**	**M(A)**	**D(A)**	**M(B)**	**D(B)**
30	90	0	0	0	0	100	50	120	87,5	0	62,5	0
30	90	5	5	1,5	1,5	100	50	120	87,5	1,397542	62,5	1,397542
30	90	10	10	3	3	100	50	120	87,5	2,795085	62,5	2,795085

The assumed probability are given by $q = 0.65$ and $q = 0.75$, respectively). The same compliance limits are applied (C=450000 for the first group, C=300000 for the second group). The modifications and the termination criteria of the penalty parameter are the same as they are in the deterministic examples.
As it is seen in Table 1 in symmetric cases the probabilistic adjoint loading are considered with the following covariant values: $\sigma_A^2 = \kappa_{1,1}^2 = 1{,}767767^2$, $\sigma_B^2 = \kappa_{2,2}^2 = 1{,}767767^2$, $\kappa_{1,2}^2 = 0.0$, $\kappa_{2,1}^2 = 0.0$ at first and secondly these values are $\sigma_A^2 = \kappa_{1,1}^2 = 3{,}535534^2$, $\sigma_B^2 = \kappa_{2,2}^2 = 3{,}535534^2$, $\kappa_{1,2}^2 = 0.0$, $\kappa_{2,1}^2 = 0.0$. The unsymmetrical cases are composed similarly (the numerical values are in Table 2). One can see that this method always produces symmetric variations.
Applying the iterative procedure presented earlier [3] the

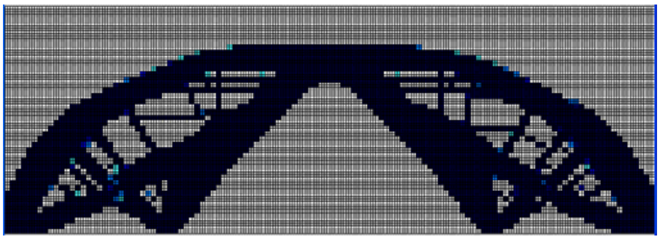

Figure 7. Numerically obtained optimal topology of the stochastic problem in symmetric case 1.

probabilistically constrained stochastic optimal topologies can be computed. It was cc. eight hundred major iteration steps to obtain the solutions.
The numerical results are presented in symmetric cases in Figure 7 and in Figure 8, respectively.
The numerically calculated optimal topologies of the unsymmetric cases are shown in Figure 9 and in Figure 10, respectively.

One can see that increasing the "level" of the uncertainties the

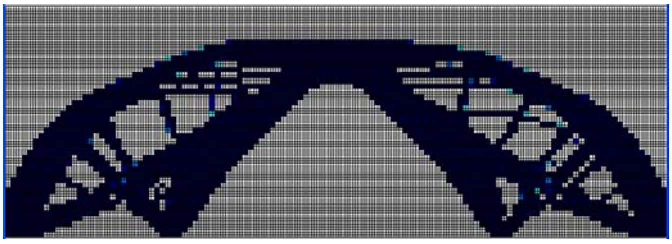

Figure 8. Numerically obtained optimal topology of the stochastic problem in symmetric case 2

resulted optimal topology is more robust.

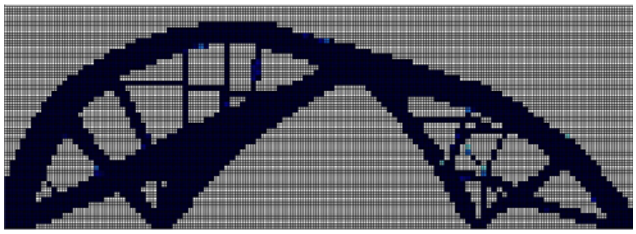

Figure 9. Numerically obtained optimal topology of the deterministic problem in unsymmetric case 1

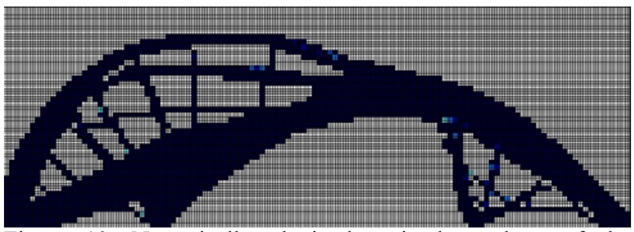

Figure 10. Numerically obtained optimal topology of the deterministic problem in unsymmetric case 2

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