A fractal stiffness model for elasto–plastic contact analysis in press joint

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Abstract

The study is undertaken to get some insight into the possibility to derive normal and tangential contact stiffness for isotropic rough surfaces. The received contact stiffness is based on fractal geometry for surface topography description using a two–variable Weierstrass–Mandelbrot function. The single microcontact stiffness is obtained by using the classical Hertz contact theory. The total contact stiffess is assumeed to be sum of individual stiffnesses corresponding to the stiffness of the spherical single asperity. The present fractal model is used to finite element elasto–plastic contact analysis in the fixed fit–press assembly.

Keywords: contact stiffness, Weierstrass–Mandelbrot function, finite elements, press joint

1. Theoretical background

Fractal geometry, pionneered by Mandelbrot [1], can be observed in various natural phenomena, is characterized by continuity, nondifferentiability and self–affinity. These mathematical properties are satisfied by the Weierstrass–Mandelbrot function given by [2,3,4]

$$z(x) = L\left(\frac{G}{L}\right)^{D-1}(\ln\gamma)^{1/2}\sum_{n=0}^{n_{max}}\gamma^{(D-2)n}\left[\cos\phi_{1n} - \cos\left(\frac{2\pi\gamma^n x}{L}\right) - \phi_{1n}\right] \quad (1)$$

where z is the surface height, x is the lateral distance, D is the fractal dimension of a surface profile ($1 < D < 2$), γ is the scaling parameter for determining the spectral density and self–affine property ($\gamma > 1$), L is the lenght of a fractal sample to be characterized and ϕ_{1n} is a random phase. The right side of equation (1) is a superposition of cosine functions with geometrically increasing frequencies. The random phase ϕ_n is used to prevent the coincidence of different frequencies at any point of the surface profile. The scaling parameter γ controls the density of the frequencies in the surface. Based on surface flatness and frequency distribution, γ is chosen to be 1.5 [2]. The fractal rouhgnessy G is a height scaling parameter independent of the frequency. A rougher surface is characterized by higher G values. The fractal dimension D determines the distribution of high–and low–frequency components in the surface profile; larger values of D correspond to denser profiles. For isotropic surfaces, the value of D can be determined from the slope of the log–log plot of the power spectral density function which can be written in the following form [2]:

$$S(\omega) = \frac{G^{2(D-1)}}{2\ln\gamma}\frac{1}{\omega^{(5-2D)}} \quad (2)$$

The spectral density function can be determined experimentally.

2. Contact modelling

When two bodies with nominally flat surfaces are brought into contact, the area of real contact is only a small fraction of the nominal contact area where asperities from one solid are squeezed against asperities from another solid. The asperities can deform elastically or plastically. It is assumed that the asperities are sufficiently apart from each other in order to avoid the interactions between them.

Assuming that for an asperity with truncated microcontact radius r', the longest wavelenght in the asperity waveform is $L = 2r'$, the asperity interference δ is then determined by the cosine term of the fractal function (1) and is equal to the peak–to–valley amplitude of the cosine function [5]

$$\delta = 2G^{(D-1)}\ln\gamma^{1/2}(2r')^{(2-D)} \quad (3)$$

The interference of a spherical asperity δ is related to its radius of curvature R as follows

$$R^2 = (R-\delta)^2 + (r')^2. \quad (4)$$

Since the asperity radius of curvature is typically orders of magnitude greater than the asperity height, the relation (4) is reduced to

$$(r')^2 = 2R\delta \quad (5)$$

To calculate the quivalent radius of curvature of the cosine–shaped asperity, its shape is approximated by a circular profile. Then, substituting the intererence δ from equation (3) into relation (5) become

$$R = \frac{(a')^{D/2}}{2^{(4-D)}\pi^{D/2}\,G^{(D-1)}\,(\ln\gamma)^{1/2}} \quad (6)$$

where a' is the truncated area of the microcontact, $a' = \pi(r')^2$. The single microcontact stiffness model is obtained by using the Hertz contact theory. From the Hertz theory the elastic normal force F_e is given by

$$F_e = \frac{4E^* r^3}{3R} \quad (7)$$

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where E^* denotes effective Young's modulus and r is the radius of the deformed contact area given by $a = \pi r^2$. For a circular elastic microcontact, $a' = 2a$, thus $a' = 2\pi r^2$. So, from equation (5) taking into account the previous relationships we have

$$r^3 = (R\delta)^{3/2} \tag{8}$$

Thus, the elastic force F_e, can be then obtained as the function of interface δ by substituting equations (6) and (8) into equation (7)

$$F_e = \frac{4}{3}E^*\delta^{3/2}\frac{(2\pi\delta)^{\frac{D}{2(2-D)}}}{2^{\frac{4-D}{2-D}}\pi^{\frac{D}{2(2-D)}}G^{\frac{D-1}{2-D}}\,(\ln\gamma)^{\frac{1}{2(2-D)}}} \tag{9}$$

The normal contact stiffness k_n at single asperity is defined as

$$k_n = \frac{\mathrm{d}F_e(\delta)}{\mathrm{d}\delta} \tag{10}$$

From equation (9) yields

$$k_n = \frac{4}{3}E^*\frac{3-D}{2-D}\delta^{\frac{1}{2-D}}\frac{2^{\frac{D}{2(2-D)}}}{2^{\frac{4-D}{2-D}}G^{\frac{D-1}{2-D}}\,(\ln\gamma)^{\frac{1}{2(2-D)}}} \tag{11}$$

which is equaivalent to the following simple formula

$$k_n = \frac{4}{3\sqrt{\pi}}E^*\left(\frac{3-D}{2-D}\right)a^{1/2} \tag{12}$$

where a denotes the area of the deformed asperity.

The tangential contact stiffness between two circular asperities is expressed as [6]

$$k_t = \frac{4Gr}{2-\nu} \tag{13}$$

where G, r and ν correspond to the shear modulus of elasticity, radius of the deformed area and Poisson ratio, respectively.

3. Contact stiffness

The total normal and tangential stiffnesses can be determined by integrating the microstiffness k_n and k_t (see, equations (12) and (13)) using the truncated asperity size distribuation function $n(a')$ which can be written as [5]

$$n(a') = \frac{D-1}{2a'_L}\left(\frac{a'_L}{a'}\right)^{(D+1)/2} \tag{14}$$

where a'_L is the size of the largest truncated microconatct at a given mean separation distance. The number of microcontacts having truncated areas between a' and $a' + \mathrm{d}a'$ is then given by $n(a')\mathrm{d}a'$. At the given mean surafce separation distance, the truncated area of the largest microcontact a'_L, can be determinated from the total area of equivalent rough surface S' by the following expression [5]

$$S' = \int_0^{a'_L} a'n(a')\mathrm{a}' \tag{15}$$

Substituting equation (14) into the above equation and integrating, become

$$S' = \left(\frac{D-1}{3-D}\right)a'_L \tag{16}$$

The total truncated area, S', can be obtained from numerical integration of the truncated areas of the rough surfaces, and a'_L can then be calculated from equation (16).

The total normal contact stiffness K_n is given by

$$K_n = \int_0^{a'_L} k_n\; n(a')\mathrm{d}a' \tag{17}$$

Substituting equations (12) and (14) into equation (17) we arrive at

$$K_n = \frac{4}{3\sqrt{2\pi}}E^*\left(\frac{(3-D)(D-1)}{(2-D)^2}\right)(a'_L)^{1/2} \tag{18}$$

In the same manner, the total tangential stiffness K_t with the help of equation (13) can be written as

$$K_t = \int_0^{a'_L} k_t\; n(a')\mathrm{d}a' = \int_0^{a'_L}\frac{4Gr}{2-\nu}\; n(a')\mathrm{d}a' \tag{19}$$

and after integrating we have

$$K_t = \frac{2G(D-1)}{(2-D)(2-\nu)\sqrt{\pi}}(a'_L)^{1/2} \tag{20}$$

The present formulae given by equations (18) and (20) are different from them given by Jiang *at al.* [7].

The normal and tangential stiffnesses have been incorporated into an axialsymmetric finite element programm for elasto–plastic contact analysis of the fit–press joint.

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