Volumetric growth of solid bodies: mechanical framework and mathematical aspects

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Abstract

General balance laws for solid bodies undergoing growth phenomena are written in the framework of irreversible thermodynamics of open systems. Growth is viewed as a multiphysical phenomenon coupling transport of nutrients with mechanical factors promoting a local variation of the density, at constant number of particles. The growing continuum body is endowed with an hyperelastic constitutive behavior. The entropy production allows the writing of a growth model relating the growth velocity gradient to an Eshelby stress. The PDE system of field equations is analyzed from a mathematical point of view, especially as to the boundedness of solutions.

Keywords: volumetric growth, balance laws, irreversible thermodynamics, configurational mechanics, multiphysical couplings, mathematical aspects

1. Introduction

The kinematics of volumetric growth is first written, assuming growth resulting from a local increase of the mass density at constant particle numbers. Regarding notations, vectors and tensors are denoted by boldface symbols. The inner product of two second order tensors is denoted $(\mathbf{A}.\mathbf{B})_{ij} = A_{ik}B_{kj}$. The kinematics of growth is elaborated from the classical multiplicative decomposition [4] of the transformation gradient

$$\mathbf{F} = \nabla_X \mathbf{x}(\mathbf{X}, t) \rightarrow J := \det(\mathbf{F}) \tag{1}$$

with $\mathbf{X}, \mathbf{x}$ the Lagrangian and Eulerian positions respectively in the referential and actual configurations denoted Ω_R,Ω_t respectively, as the product of the growth deformation gradient $\mathbf{F}_g$ and the growth accommodation mapping $\mathbf{F}_a$

$$\mathbf{F} = \mathbf{F}_a.\mathbf{F}_g \tag{2}$$

The transformation gradients $\mathbf{F}_a, \mathbf{F}_g, \mathbf{F}$ define the mappings of the tangent spaces to the various configurations. The Jacobean of the growth mapping informs about the nature of growth:

$$J_g := \det(\mathbf{F}_g) \tag{3}$$

Hence $J_g > 1$ describes growth, whereas $J_g < 1$ represents resorption. Growth (or resorption) essentially occurs between the referential and the actual configurations, and is traduced by a local change of density.

2. Mass balance

The mass balance in global form expresses in Eulerian description as

$$\begin{aligned} \frac{D}{Dt}\int\limits_{\Omega(t)} \rho dx &= \int\limits_{\Omega(t)} \pi dx - \int\limits_{\partial\Omega(t)} \{\rho \mathbf{v}_g + \mathbf{J}(\rho)\}.\mathbf{n} dA = \\ &= \int\limits_{\Omega(t)} \pi dx + \int\limits_{\partial\Omega(t)} \phi(\rho).\mathbf{n} dA = \int\limits_{\Omega_t} \Gamma \rho dx \end{aligned} \tag{4}$$

in accordance with [2], denoting therein ρ the actual density, $\mathbf{v}_g$ the growth velocity field, defined as the relative velocity of material points with respect to the moving boundary $\partial\Omega(t)$ (which may have its own geometrical velocity), $\phi(\rho)$ the total mass flux, and π the mass source term. The last equality defines the growth rate of mass as

$$\Gamma = \dot{J}_g / J_g = Tr(\mathbf{D}_g) \tag{5}$$

with $\mathbf{D}_g := \dot{\mathbf{F}_\mathbf{g}}.\mathbf{F}_\mathbf{g}^{-1}$. Using the convective derivative along an arbitrary velocity field $\mathbf{w}$ of any quantity $a = a(\mathbf{x}, t)$, elaborated as

$$\frac{\delta_w a}{\delta t} = \left(\frac{\partial a}{\partial t}\right)_x + \nabla a.\mathbf{w}, \tag{6}$$

the strong form of the mass balance results as

$$\frac{D\rho}{Dt} = \pi + \nabla.\phi(\rho) - \rho \nabla.\mathbf{v} \tag{7}$$

Adopting the framework of hyperelasticity, the first Piola-Kirchhoff stress $\mathbf{P}$ expresses from the strain energy density per unit volume in the reference configuration $W(\mathbf{F};\mathbf{X})$ (a possible explicit dependence upon the Lagrangian variable is included for heterogeneous media) as

$$\mathbf{P} := \partial_\mathbf{F} W(\mathbf{F}_a; \mathbf{X}) \tag{8}$$

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The balance of momentum expresses in Eulerian description and using the mass balance as

$$\rho\frac{D\mathbf{v}}{Dt}=\mathbf{f}+div\sigma+(\phi.\nabla)\,\mathbf{v} \tag{9}$$

with σ the Cauchy stress, such that $\mathbf{P}:=J\sigma.\mathbf{F}^{-t}$.

3. Local dissipation and growth model

The global form of Clausius-Duhem expresses in Eulerian description as [3]:

$$\int_{\Omega}\rho\dot{\psi}dx\leq -P_i+\underline{J}_k.\underline{F}_k+\Phi_m \tag{10}$$

with $\Phi_m:=-\int_{\partial\Omega}\mu_i\mathbf{J}_i.\mathbf{n}d\sigma$

the flux of mass through the boundary of Ω, P_i the virtual power of internal forces. The balance of mass accounting for the biochemical contribution expresses successively in weak and strong forms as

$$\begin{aligned}&\frac{D}{Dt}\int_{\Omega_t}\rho dx=\int_{\Omega_R}Tr\left(\dot{\mathbf{F}}_g.\mathbf{F}_g{}^{-1}\right)\rho JdX=\\&\int_{\Omega_R}\rho J\dot{n}_kdX+\int_{\partial\Omega_R}\mathbf{J}_k.\mathbf{NdA}\\&\rightarrow\rho J\dot{n}_k+Div\mathbf{J}_k=\rho J\left(\dot{\mathbf{F}}_g.\mathbf{F}_g{}^{-1}:\mathbf{I}\right)=\Gamma\rho J\end{aligned} \tag{11}$$

The last equality results from (5). The Coleman-Noll procedure then leads to the constitutive equations for the first Piola-Kirchhoff stress and the chemical potential

$$\mathbf{T}=\rho J\frac{\partial\psi}{\partial\mathbf{F}_a}.\mathbf{F}_g{}^{-t};\mu_k=\frac{\partial\psi}{\partial n_k} \tag{12}$$

The residual dissipation then splits into the sum of a *chemical dissipation* and the *intrinsic (mechanical) dissipation*

$$0\leq-\mathbf{J}_iGrad\mu_i+\rho J\left(\mathbf{F}_a{}^t.\frac{\partial\psi}{\partial\mathbf{F}_a}-(\psi+\mu_k)\,\mathbf{I}\right):\mathbf{L}_g \tag{13}$$

Following Curie principle, one is then entitled to write a general growth model according to

$$\mathbf{L}_g=f\left(\tilde{\mathbf{\Sigma}}_a\right) \tag{14}$$

with the Eshelby stress given by, see [1]

$$\tilde{\mathbf{\Sigma}}_a:=\rho\mathbf{F}_a{}^t.\frac{\partial\psi}{\partial\mathbf{F}_a}-\rho\,(\psi+\mu_k)\,\mathbf{I} \tag{15}$$

4. Local dissipation and growth model

Considering as a first step a purely mechanical situation (the source and flux of mass are lumped into global contributions, not relating as before those contributions to biochemical processes), the system of PDEs under consideration writes:

$$\begin{cases}\frac{\partial}{\partial t}(\rho J)=\Gamma\rho J\\ div\mathbf{P}=-\rho J\mathbf{f}\\ \mathbf{P}=\frac{\partial W}{\partial\mathbf{F}_a}\rightarrow\mathbf{\Sigma}_a=WI-\mathbf{F}_a{}^t.\mathbf{P}\\ \frac{\partial\mathbf{F}_g}{\partial t}.\mathbf{F}_g{}^{-1}=f\left(\mathbf{\Sigma}_a\right)\end{cases} \tag{16}$$

with

$$f\left(\mathbf{\Sigma}_a\right)=f_0\left(I_i\right)\mathbf{I}+f_0\left(I_i\right)\mathbf{\Sigma}_a+f_0\left(I_i\right)\mathbf{\Sigma}_a{}^2 \tag{17}$$

acccording to the representation of isotropic functions with tensorial argument, with I_i the principal invariants of Eshelby stress. In the isotropic case, the growth mapping is isotropic, $\mathbf{F}_g=g\mathbf{I}$, with g a scalar function, hence $\mathbf{F}_a=g^{-1}\nabla\mathbf{u}$, and the mass growth rate is $\Gamma=3\dot{g}/g$. The previous system then writes

$$\begin{cases}\frac{1}{J}\frac{\partial}{\partial t}(\rho J)=\frac{3}{g}\frac{\partial g}{\partial t}\Rightarrow\rho J=C(x)\,g^3=\frac{\rho_0J_0}{g_0{}^3}g^3\\ div\frac{\partial W\left(g^{-1}\nabla\mathbf{u}\right)}{\partial\mathbf{F}_a}+\rho J\mathbf{f}=\mathbf{0}\\ \frac{\partial g}{\partial t}=f\left(g^{-1}\nabla\mathbf{u}\right)g\\ g_{|t=0}=g_0\end{cases} \tag{18}$$

The form of the solution to this BVP is studied in the 1D case, according to the possible form of the function $f\left(g^{-1}\nabla\mathbf{u}\right)$.

Growth may be unbounded, for instance when $f(.)=Cte$, since the growth function g is then solution of the ODE $\dot{g}=g^2$. *Mathematical problem.* We present the mathematical treatment of the simplest thermomechanical growth model in the isotropic case under the following assumption: the displacement vector field $\mathbf{u}(X,t)$ and the distortion tensor $\mathbf{F}(X,t)=\nabla\mathbf{u}(X,t)$ are functions of the reference frame coordinate $X\in\Omega\mathbf{R}^d$ and the temporal variable $t\in(0,T)$; $\mathbf{F}$ has the representation $\mathbf{F}=\mathbf{F}_a.\mathbf{F}_g$, where $\mathbf{F}_a$ is the part of the distortion tensor corresponding to elastic displacements, and the tensor F_g is responsible for the growth. Next we assume that $F_g=g\mathbf{I}$. Hence the evolution of the material is described by the state variables the displacement field $\mathbf{u}(X,t)$, the growth variable $g(X,t)$, and the temperature distribution $\theta(x,t)$. we restrict our considerations by quasistatic case and neglect the inertial forces. Thus we come to the system of governing equations in the form

$$\begin{aligned}&\text{div}\left(\frac{\partial W(g^{-1}\nabla\mathbf{u},\vartheta)}{\partial\mathbf{F}_a}\right)=\mathbf{f}\ \text{ in }\ \Omega\times(0,T),\\&\partial_tg=f(\nabla\mathbf{u},g,\theta)\ \text{ in }\ \Omega\times(0,T),\\&\partial_t\theta-\kappa\Delta\theta=D(g,\nabla u,\theta)\ \text{ in }\ \Omega\times(0,T),\end{aligned} \tag{19}$$

$$\partial_n\theta-\lambda\theta=0,\quad\frac{\partial W(g^{-1}\nabla\mathbf{u},\vartheta)}{\partial\mathbf{F}_a}=0\ \text{ on }\ \partial\Omega\times(0,T) \tag{20}$$

$$\theta=\theta_0,\quad g=g_0\ \text{ in }\ \omega\times\{t=0\}. \tag{21}$$

Here f and g are given functions, subjected to the Clausius-Duhem inequality, the tensor derivative is defined by the relation

$$\frac{\partial W(\mathbf{F}_a,\vartheta)}{\partial\mathbf{F}_a}:\mathbf{H}=:\lim_{\epsilon\to0}\frac{1}{\epsilon}\Big\{W(\mathbf{F}_a+\epsilon\mathbf{H},\vartheta)-W(\mathbf{F}_a,\vartheta)\Big\}. \tag{22}$$

We consider regularization of system (19), with ordinary differential equation for g replaced by the parabolic equation

$$\partial_tg=f(\nabla\mathbf{u},g,\theta)+\varepsilon\Delta\theta. \tag{23}$$

Under natural assumptions on the functions f and D, we prove the existence of weak solution for the regularized boundary value problem and investigate the singular limit as $\varepsilon\to0$.

References

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