A modified fractional step approach

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Abstract

We propose here a methodology for improving mass-conservation features of the fractional-step schemes on moving grids applied to viscous incompressible flows containing free surfaces. The idea is to find a better approximation for the intermediate velocity. This is achieved by using a prediction upon the end-of-step pressure in the pressure gradient term of the fractional momentum equation. Thus, an intermediate velocity results being much closer to the final one, than that of the standard fractional step scheme. We propose a methodology, where this goal can be achieved without necessitating to resolve any additional linear system.

Keywords: fractional step method, PFEM, incompressible fluid, Lagrangian methods, CFD, mass conservation

1. Introduction

Fractional step method for the incompressible Navier-Stokes equations gained its popularity due to its computational efficiency [1], [2], achieved by decoupling of the velocity and the pressure. However its use introduces several drawbacks.

A well-known difficulty in the application of fractional step methods to the simulation of the free-surface flows is the conservation of mass. When using mesh-moving methods (ALE or Lagrangian, e.g. PFEM [3]) this artefact becomes apparent: the fluid volume diminishes within the simulation.

The main problem originates from the necessity of using an "artificial" homogeneous Dirichlet b.c. for the pressure at the free surface, when using classical continuous Laplacian approximations in the pressure Poisson' equation. While using a discrete Laplacian instead, improves the situation, the computational effort grows considerably thus making this option rarely used in practice.

Additionally, fractional step method does not work well for large time steps. In other words, the error grows as the pressure that is used in the fractional momentum equation differs from the "true" end-of-step one. This error is known to be of first order in time when fractional momentum equation does not contain pressure gradient term at all. The error is of second order when the previous time step pressure is assumed [4].

Here we propose a methodology that enables to cure both these problems, i.e. mass loss and precision in resolution of the convection. The idea is to use a prediction upon the end-of-step pressure.The objective is to use such an approximation that should not involve resolution of any equations system, thus not increasing the computational effort considerably. Including the approximated end-of-step pressure in the fractional momentum equation results in the intermediate velocity being much closer to the divergence-free one. Thus, the convection can be resolved much better.

As in the classical fractional step method, once the fractional velocity is computed, the Poisson's equation is solved for the pressure. The Poisson' equation in our technique serves as a corrector acting upon the predicted pressure field to give the final incompressible end-of-step pressure. While correcting the pressure within the domain, we propose to keep the predicted pressure at the free surface. This means that the approximated pressure serves as a Dirichlet boundary condition necessary for resolving the Poisson' equation when continuous Laplacian approximation is used. Thus the "artificial" zero-pressure boundary condition at the free surface is never used.

Having computed the end-of-step pressure, the correction step is applied in a standard manner and the incompressible velocity end-of-step velocity is obtained.

The paper is organized as follows. We first introduce the modified fractional equations. For the fractional momentum equation we introduce the approximation for the end-of-step pressure and its linearization. In the summary section we highlight the advantages of the modified method. All the developments of the present work are illustrated in Lagrangian framework.

2. Modified fractional step

The discrete momentum-continuity system (for simplicity we use BE scheme for the illustration; the details upon the space discretization are omitted) for the incompressible flow in the updated Lagrangian framework can be written as:

$$\mathbf{r}_m = \mathbf{F} - \left(\rho\mathbf{M}\frac{\bar{\mathbf{v}}_{n+1} - \bar{\mathbf{v}}_n}{\Delta t} + \mu\mathbf{L}\bar{\mathbf{v}}_{n+1} + \mathbf{G}\bar{p}_{n+1}\right) \tag{1}$$

$$\mathbf{D}\bar{\mathbf{v}}_{n+1} = 0 \tag{2}$$

where $\mathbf{r}_m$ is the momentum equation residual[1], $\mathbf{M}$ is the mass matrix, $\mathbf{L}$ is the Laplacian matrix, $\mathbf{G}$ is the gradient matrix, $\bar{\mathbf{v}}$, $\bar{p}$ and μ are the velocity, pressure and viscosity respectively and $\mathbf{F}$ is the body force vector.

Next we introduce the modified fractional step split, that consists in using a pressure prediction in the fractional momentum equation:

$$\tilde{\mathbf{r}}_m = \mathbf{F} - \left(\rho\mathbf{M}\frac{\tilde{\mathbf{v}} - \bar{\mathbf{v}}_n}{\Delta t} + \mu\mathbf{L}\tilde{\mathbf{v}} + \mathbf{G}\bar{p}^g_{n+1}\right) \tag{3}$$

$$\rho\mathbf{M}\frac{\bar{\mathbf{v}}_{n+1} - \tilde{\mathbf{v}}}{\Delta t} + \mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^g_{n+1}\right) = 0 \tag{4}$$

$$\mathbf{D}\bar{\mathbf{v}}_{n+1} = 0 \tag{5}$$

The pressure Poisson equation is obtained by applying the incompressibility condition 5 to the end-of-step momentum equa-

[1]Note that writing the equations in the updated Lagrangian framework, i.e. in the unknown configuration $\mathbf{x}_{n+1}$ makes the discrete operators non-linear and obliges us to use residual form.

tion, leading to

$$\mathbf{D}\tilde{\mathbf{v}} = \Delta t \mathbf{D}\mathbf{M}^{-1}\mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^{g}_{n+1}\right) \tag{6}$$

For simplicial equal interpolation order mixed elements, pressure needs to be stabilized. This consists in adding a stabilization term into the pressure equation. Indicating the stabilization term (we do not specify here the particular stabilization technique) as $\tau\mathbf{S}$ and using the approximation $\mathbf{D}\mathbf{M}^{-1}\mathbf{G} \approx \mathbf{L}$, we arrive at the final system:

$$\tilde{\mathbf{r}}_{mom} = \mathbf{F} - \left(\mathbf{M}\frac{\tilde{\mathbf{v}} - \bar{\mathbf{v}}_n}{\Delta t} + \mu\mathbf{L}\tilde{\mathbf{v}} + \mathbf{G}\bar{p}^{g}_{n+1}\right) \tag{7}$$

$$\mathbf{D}\tilde{\mathbf{v}} = \Delta t \mathbf{L}\left(\bar{p}_{n+1} - \bar{p}^{g}_{n+1}\right) + \tau\mathbf{S}\bar{p}_{n+1} \tag{8}$$

$$\mathbf{M}\frac{\bar{\mathbf{v}}_{n+1} - \tilde{\mathbf{v}}}{\Delta t} + \mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^{g}_{n+1}\right) = 0 \tag{9}$$

2.1. Fractional momentum equation solution

To solve the fractional momentum equation, the pressure prediction $\bar{p}^g$ shall be computed. Note that assuming that it is equal to zero ($\bar{p}^g = 0$) or to the previous step pressure ($\bar{p}^g = \bar{p}_n$) would recover standard first and second order fractional step schemes respectively. In this work we propose to obtain the prediction by associating the pressure increment with the volume change as (this way of computing pressure is presented in detail in [5]), equivalent to assuming the slight compressibility.

$$\delta\bar{p}^I = \kappa\frac{\delta V^I}{V^I_n} \tag{10}$$

where $\bar{p}^I$ is the pressure at the node I, κ is the compressibility constant and V^I_n is the nodal volume (we associate to each node of the FE mesh a nodal volume defined in such a way that it coincides with the diagonal entry of the diagonalized pressure mass matrix $\bar{V}_I := \mathbf{M}_{p,II}$).

Application of Newton's method to the solution of the non-linear fractional momentum equation (Eq. 7) requires to evaluate the dynamic tangent matrix $\mathbf{H}$:

$$\mathbf{H} = \frac{\partial\mathbf{r}^{frac}_m}{\partial\mathbf{v}} = \frac{\mathbf{M}}{\Delta t^2} + \mu\mathbf{L} + \kappa\mathbf{G}\mathbf{M}^{-1}\mathbf{D} \tag{11}$$

where the last term corresponds to the linearization of the pressure gradient (see [5]).

The iterative solution of the fractional momentum equation can be implemented in the following way:

1. Solve $\mathbf{H}d\tilde{\mathbf{v}} = \tilde{\mathbf{r}}_m$
2. Update pressure according to 10 and add it to the fractional momentum residual
3. Update the discrete operators according to the new nodal position
4. Repeat until convergence in terms of velocity is achieved

We see, that the tangent of the fractional momentum equation contains the volumetric term, i.e. linearization of the pressure gradient. The additional computational cost due to the pressure update is minimal, as it does not involve solution of any system, provided that the unknown pressure is multiplied by the lumped pressure mass matrix. The cost of adding the $\kappa\mathbf{G}\mathbf{M}^{-1}\mathbf{D}$ term to the tangent matrix can be minimized by using a matrix-free method (see [5]).

2.2. Pressure Poisson' equation and the correction step

The next step to be carried out is the correction of the pressure, i.e. obtaining the end-of-step incompressible pressure. This is done by solving Eq. 8. Solution of Eq. 8 requires to impose the pressure boundary condition. While usually zero pressure is fixed at the free surface, we propose to use a physically more meaningful option, i.e. the predicted pressure. The predicted "quasi-incompressible" pressure is a meaningful physical approximation, provided that the compressibility constant κ used in the pressure prediction is large.

This step can be thus viewed as a correction of the predicted pressure $\bar{p}^g_{n+1}$ to the correct end-of-step one everywhere except for the free surface, where the predicted pressure is kept as a Dirichlet b.c..

The correction step is carried out according to Eq. 9 and returns the end-of-step divergence-free velocity.

3. Summary and future work

The proposed technique bases on approximating the pressure in the momentum equation using a prediction that assumes slight compressibility. The pressure, however is corrected by solving the pressure Poisson's equation to give the final incompressible one. The resulting combination of a quasi-incompressible prediction with a fractional step split permitted us to achieve the following advantages:

1. Much better mass conservation than that of the classical fractional step
2. Better resolution of convection, as the intermediate velocity approximates the end-of-step velocity by using a prediction upon the end-of-step pressure in the momentum equation. This, in principle, allows to use larger time steps.
3. "Natural" Dirichlet boundary condition at the free surface
4. Truely incompressible solution, as the quasi-incompressible assumption is used only in predicting the velocity and pressure, both of which are corrected afterwards

A crucial aspect, that defines future research line is the estimation of the optimal value of the compressibility constant used in the pressure prediction. On one hand, it should be sufficiently large to insure that the pressure wave propagates in the computational domain within one time step, on the other hand it should not be too large to spoil the system conditioning and make it "too stiff" for the linear solvers.

Another line of research clearly originiating from this work is the application of the modified fractional step approach in the context of the fluid-structure interaction. The presence of the pressure prediction in the fractional momentum might have an important impact upon the convergence of the FSI iterations.

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