Boundary behavior of viscous fluids

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Abstract

We consider a family of solutions to the evolutionary Navier-Stokes system supplemented with the complete slip boundary conditions on domains with rough boundaries. We give a complete description of the asymptotic limit by means of $\Gamma-$convergence arguments, and identify a general class of boundary conditions.

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1. Introduction

A proper choice of boundary conditions plays a significan role in mathematical flui mechanics. In the case of an impermeable boundary, meaning,

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1}$$

where $\mathbf{u}$ denotes the velocity of the flui and $\mathbf{n}$ stands for the outer normal vector to the boundary of a spatial domain $\Omega \subset R^3$ occupied by the fluid a commonly accepted hypothesis asserts that the flui adheres completely to the boundary. If the latter is at rest, such a stimulation gives rise to the *no-slip boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{2}$$

Although the no-slip hypothesis seems to be in a good agreement with experiments, it leads to certain rather surprising conclusions, the most striking one being the absence of collisions between a rigid body immersed in a linearly viscous flui and the boundary $\partial\Omega$ (see Hesla [2] Hillairet [3]). In contrast with (2), the so-called *Navier's boundary condition*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ S\mathbf{n} + \beta\mathbf{u}|_{\partial\Omega} = 0, \tag{3}$$

where S is the viscous stress tensor, offer more freedom and are likely to solve several paradoxical phenomena resulting from the no-slip boundary conditions.

We consider the Navier-Stokes system

$$\left\{ \begin{array}{rcl} \partial_t \mathbf{u}_\varepsilon + \text{div}\,(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu \text{div}\,\mathbf{D}[\mathbf{u}_\varepsilon] + \nabla p_\varepsilon &=& \mathbf{g} \\ \text{div}\,\mathbf{u}_\varepsilon &=& 0 \\ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} &=& 0 \\ (\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}_\varepsilon)_{tan}|_{\partial\Omega_\varepsilon} &=& 0 \\ \mathbf{u}_\varepsilon(0,\cdot) &=& \mathbf{u}_0, \end{array} \right\} \tag{4}$$

$\nu > 0$,

where we have set

$$\mathbf{D}[\mathbf{u}_\varepsilon] = \frac{1}{2}(\nabla \mathbf{u}_\varepsilon + \nabla^t \mathbf{u}_\varepsilon).$$

Our goal is to investigate the asymptotic behavior of solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ for $\varepsilon \to 0$. It is easy to check that the functions $\mathbf{u}_\varepsilon$, suitably extended to the whole set D converge to a limit $\mathbf{u}$ that satisfie equation (4) on the domain Ω. Consequently, the principal issue o be discussed in this paper is identifying the boundary conditions for $\mathbf{u}$. We shall see that $\mathbf{u}$ satisfie the so-called *friction-driven* boundary conditions, where eat each point of the boundary a certain (possibly empty) component of the velocity vanishes while its complementary part satisfie a kind of the Navier boundary conditions specifie in (3) for a suitable function β. See [1].

2. Analysis of Stokes system with friction-driven boundary conditions

Given $\mathbf{f} \in L^2(D, \mathbb{R}^N)$, $N = 2, 3$, we consider a (perturbed) Stokes problem in the form:

$$\left\{ \begin{array}{rcl} -\text{div}\,\mathbf{D}[\mathbf{u}_\varepsilon] + \mathbf{u}_\varepsilon + \nabla p_\varepsilon &=& \mathbf{f} \text{ in } \Omega_\varepsilon \\ \text{div}\,\mathbf{u}_\varepsilon &=& 0 \text{ in } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon \cdot \mathbf{n} &=& 0 \text{ on } \partial\Omega_\varepsilon \\ (\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}_\varepsilon)_{tan} &=& 0 \text{ on } \partial\Omega_\varepsilon \end{array} \right\} \tag{5}$$

Problem (5) is understood in the variational sense, where the solutions are regarded as minimizers of the associated energy functional

$$\mathbf{v} \mapsto \frac{1}{2} \int_{\Omega_\varepsilon} |\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2 \ dx - \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{v} dx \tag{6}$$

in the class

$$\mathbf{v} \in H^1(\Omega_\varepsilon; \mathbb{R}^N),\ \text{div}\,\mathbf{v} = 0 \text{ in } \Omega_\varepsilon,\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7}$$

Let μ be a finit positive Borel measure concentrated on $\partial\Omega$ and absolutely continuous with respect to capacity. In addition, we are given a family of linear spaces $\mathcal{V} := \{V(x)\}_{x\in\partial\Omega}$, where $V(x)$ is a subspace of the tangent hyperplane at $x \in \partial\Omega$, in particular, the dimension of $V(x)$ does not exceed $N-1$. Furtermore, let $a_{i,j} : \partial\Omega \to \mathbb{R}$, $1 \le i,j \le N$, be Borel functions such that $a_{i,j} = a_{j,i}$, and $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \ge 0$ for all $\xi \in \mathbb{R}^N$. We set $A = \{a_{i,j}\}_{i,j=1}^N$. Finally, we take $\mathbf{f} \in L^2(D,\mathbb{R}^N)$.

We say that $\mathbf{u}$ is a solution to Stokes system with friction-driven boundary conditions determined by means of the trio $\{\mu, A, \mathcal{V}\}$ if $\mathbf{u}$ solves the minimization problem

$$\mathcal{J}(\mathbf{u}) = \min_{\mathbf{v}\in\mathcal{C}} \mathcal{J}(\mathbf{v}), \tag{8}$$

where

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}\int_\Omega \ |\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2 \ \ dx + \frac{1}{2}\int_{\partial\Omega} A\mathbf{v}\cdot\mathbf{v} d\mu - \int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \tag{9}$$

and

$$\mathcal{C} := \Big\{ \mathbf{v} \in H^1(\Omega,\mathbb{R}^N) \quad \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega, \\ ; \mathbf{v}(x) \in V(x) \text{ for q. a. } x \in \partial\Omega \Big\}.$$

As $\mathcal{C}$ is a closed subspace of $H^1(\Omega,\mathbb{R}^N)$, the classical Lax-Milgram theorem together with Korn's inequality yield the following result.

Theorem 2.1 *Problem* (8) *has a unique solution.*

Remark 2.2 Writing the Euler equation associated to the minimization problem (8) we *formally* get

$$\begin{cases} -\mathrm{div}\,\mathbf{D}[\mathbf{u}] + \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega \\ \mathrm{div}\,\mathbf{u} &= 0 \text{ in } \Omega \\ \mathbf{u}(x) &\in V(x) \text{ for q.e. } x \in \partial\Omega \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big]\cdot\mathbf{v} &= 0 \text{ for any } \mathbf{v} \in V(x),\ x \in \partial\Omega. \end{cases} \tag{10}$$

The *driven* part of the boundary conditions is given by the family of spaces $\{V(x)\}_{x\in\partial\Omega}$ while the *friction* part is determined by the matrix $A = \{a_{i,j}\}_{i,j=1}^N$ and the measure μ.

Here is the main result of the section.

Theorem 2.3 *Let Ω_ε converge to Ω for more details see [1], and let $\mathbf{f} \in L^2(D,\mathbb{R}^N)$ be given. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be the family of (weak) solutions to problem* (5) *in Ω_ε.*

Then, at least for a suitable subsequence,

$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega\mathbf{u}$ *(strongly) in* $L^2(D,\mathbb{R}^N)$,

$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega\nabla\mathbf{u}$ *weakly in* $L^2(D,\mathbb{R}^{N\times N})$,

where $\mathbf{u}$ is a solution of the minimization problem (8), or, equivalently, a weak solution of (10) in Ω, for a suitable trio $\{\mu, A, \mathcal{V}\}$ independent of the driving force $\mathbf{f}$.

3. Navier-Stokes system

We say that $\mathbf{u}_\varepsilon$ is a weak (variational) solution to problem (4) if

$$\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon,\mathbb{R}^N) \cap L^2(0,T;H^1(\Omega_\varepsilon,\mathbb{R}^N)), \\ \mathbf{u}_\varepsilon\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \mathrm{div}\,\mathbf{u}_\varepsilon = 0, \tag{11}$$

and the integral identity

$$\int_0^T\int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon\cdot\partial_t\varphi + \mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla\varphi - \nu\mathbf{D}[\mathbf{u}_\varepsilon] : \mathbf{D}[\varphi]) \ dx\ dt \tag{12}$$

$$= -\int_{\Omega_\varepsilon} \mathbf{u}_0\cdot\varphi(0,\cdot)\ dx - \int_0^T\int_{\Omega_\varepsilon} \mathbf{g}\cdot\varphi\ dx\ dt$$

holds for any test function φ such that $\varphi \in C_c^1([0,T)\times\overline{\Omega}_\varepsilon;\mathbb{R}^N)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$, $\mathrm{div}\,\varphi = 0$.

In addition, we focus on the class of *turbulent* weak solutions in the sense of Leray that satisfy the *energy inequality*

$$\int_{\Omega_\varepsilon}\tfrac{1}{2}|\mathbf{u}_\varepsilon|^2(\tau,\cdot)\ dx + \nu\int_0^\tau\int_{\Omega_\varepsilon}|\mathbf{D}[\mathbf{u}_\varepsilon]|^2\ dx\ dt \le \\ \int_{\Omega_\varepsilon}\tfrac{1}{2}|\mathbf{u}_0|^2\ dx + \int_0^\tau\int_{\Omega_\varepsilon}\mathbf{g}\cdot\mathbf{u}_\varepsilon\ dx\ dt \tag{13}$$

for a.a. $\tau \in (0,T)$.

Assuming $\mathbf{u}_0 \in L^2(D,\mathbb{R}^N)$, and, say, $\mathbf{g} \in L^\infty(0,T;L^2(D,\mathbb{R}^3))$, we may infer from (13) that $\mathbf{u}_\varepsilon \to \mathbf{u}$ weakly in $L^2(0,T;H^1(D,\mathbb{R}^N))$, and $\mathbf{u}_\varepsilon \to \mathbf{u}$ weakly-(*) in $L^\infty(0,T;L^2(D,\mathbb{R}^N))$ provided $\mathbf{u}_\varepsilon$ were extended on the set D.

Finally the following theorem holds:

Theorem 3.1 *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains in $D \subset \mathbb{R}^N$, $N = 2,3$, satisfying the uniform cone condition. Let*

$$\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon,\mathbb{R}^N)) \cap L^2(0,T;H^1(\Omega_\varepsilon,\mathbb{R}^N))$$

be a sequence of weak solutions to the Navier-Stokes system (4) satisfying the energy inequality (13).

Then there exists a trio $\{\mu, A, \mathcal{V}\}$ such that, at least for a suitable subsequence,

$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega\mathbf{u}$ *weakly-(*) in* $L^\infty(0,T;L^2(D,\mathbb{R}^N))$,

$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega\nabla\mathbf{u}$ *weakly in* $L^2(0,T;L^2(D,\mathbb{R}^{N\times N}))$,

where $\mathbf{u}$ is a weak solution of problem

$$\begin{cases} \partial_t\mathbf{u} + \mathrm{div}(\mathbf{u}\otimes\mathbf{u}) - \nu\mathrm{div}\mathbf{D}[\mathbf{u}] + \nabla p &= \mathbf{g} \textit{ in } (0,T)\times\Omega \\ \mathrm{div}\,\mathbf{u} &= 0 \textit{ in } (0,T)\times\Omega \\ \mathbf{u}(x) &\in V(x) \textit{ for q.a. } x \in \partial\Omega \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big](x)\cdot\mathbf{v} &= 0 \textit{ for q.a. } x \in \partial\Omega,\ \mathbf{v} \in V(x) \\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0 \textit{ in } \Omega \end{cases}$$

in the sense specifie in (11 - 12). Moreover, $V(x) \perp \mathbf{n}(x)$ for any $x \in \partial\Omega$.

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