Application of XFEM with shifted-basis approximation to stress intensity factors computation

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Abstract

In the paper, the extended finite element method (XFEM) is applied to compute stress intensity factors (SIFs) in a two-dimensional cracked body. The XFEM is an effective tool for analyzing problems with singular fields, e.g. fracture mechanics problems. It is based on the idea of enriching the approximation in the vicinity of the discontinuity. As a result, the numerical model consists of three types of elements: standard finite elements, fully enriched elements (the domain of which is cut by a discontinuity), and partially enriched elements (the so-called *blending elements*). In a blending element, some but not all of the nodes are enriched, which adds to the approximation parasitic term. The error caused by the parasitic terms is partly responsible for the degradation of the convergence rate. It also limits the accuracy of the method. Eliminating blending elements from approximation space and replacing them with standard elements, together with applying shifted-basis enrichment makes it possible to avoid the problem. The numerical examples show improvements in results when compared with the standard XFEM approach.

Keywords: numerical analysis, finite element method, cracks

1. Introduction

The extended finite element method, first introduced by Belytschko and Black [1] and Möes *et al.* [5], provides an effective tool for analyzing discontinues and singular problems. It operates through augmenting standard finite element approximation with additional functions containing the information about the solution. In the XFEM, the enriched domain is limited only to the vicinity of a crack, and unlike in the standard finite element method, the mesh does not have to align with a discontinuity. However, the method is not free from drawbacks. The main source of error arises from unwanted terms in blending elements, which are partially enriched, thus the enrichment function can not form a partition of unity. Chessa *et al.* [2] eliminated the parasitic terms by applying the assumed strain method in blending element. D.o.f. gathering technique and higher order elements have also been shown to improve the accuracy in the blending elements, Laborde *et al.* [4]. Tarancòn *et al.* [6] proposed enhanced blending elements with a fixed enrichment area. A linearly decreasing weight function over transition layer eliminates the partially enriched elements, so that partition of unity property can be satisfied everywhere, Fries [3].

In the paper, the shifted-basis enrichment (Ventura *et al.* [7], Zi *et al.* [9]) is implemented in order to make the d.o.f. of a node in a fully enriched element the physical solution of the nodal displacement. Thus, the approximation in blending elements can be reduced to standard finite element approximation as long as the enrichment function is the same or of a lower order than the partition of the unity function. Nevertheless, for higher order functions, the following modifications with removing the enriched parts in blending elements provide an improvement in computed values of stress intensity factors. To extract the SIFs, a well known J-integral and the domain form of the interaction integral, Yau *et al.* [8], are applied. For the purposes of the analysis, a corresponding MATLAB program is developed. A classic problem of fracture mechanics is used as a benchmark.

In Section 2, the model problem considered is defined, which is relative to the equilibrium of a cracked body in plane linear elasticity. Section 3 describes XFEM shifted-basis approximation for the problem. The results of the test for the benchmark problem are presented in Section 5.

2. Problem formulation

A linear elasticity problem is addressed, with isotropic homogeneous material on a two-dimensional cracked domain Ω. On the part of the boundary Γ_D the Dirichlet condition is prescribed, and the Neumann condition is defined on Γ_N. The crack Γ_C is assumed to be traction free. The boundary Γ is composed of the sets Γ_D, Γ_N and Γ_C, such that $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$. In order to describe the asymptotic field near the crack tip, local polar coordinates (r, θ) defined at the crack tip are introduced (Fig. 1), where r denotes the distance from the crack tip.

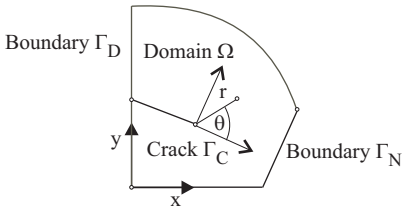

Figure 1: A cracked body with polar co-ordinates relative to the crack tip

The equilibrium of the body is expressed by the virtual work principle:

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}) d\Omega = \int_{\Omega} \boldsymbol{p} \cdot \boldsymbol{w} d\Omega + \int_{\Gamma_N} \boldsymbol{t} \cdot \boldsymbol{w} ds \tag{1}$$

where

$\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{w})$ – stress and strain tensors,

$\boldsymbol{p}, \boldsymbol{t}$ – force densities on Ω and Γ_N, respectively.

The constitutive relation is given by the Hooke's law:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{2}$$

where $\boldsymbol{C}$ is the Hooke tensor.

In the present study, small strains and displacements are considered, expressed by the strain-displacement relation:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bigtriangledown(\boldsymbol{u}) \tag{3}$$

where $\bigtriangledown$ is the symmetric part of the gradient operator.

3. Shifted-basis approximation

The XFEM approximation consists of a standard finite element part $\boldsymbol{u}(\boldsymbol{x})_C$ and the enriched part $\boldsymbol{u}(\boldsymbol{x})_E$ that enables the approximation to capture solution characteristic in the vicinity of the crack

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})_C + \boldsymbol{u}(\boldsymbol{x})_E \tag{4}$$

where

$$\boldsymbol{u}(\boldsymbol{x})_C = \sum_{J \in N_C} N_J(\boldsymbol{x}) \boldsymbol{q}_J \tag{5}$$

The enrichment term $\boldsymbol{u}(\boldsymbol{x})_E$ combines the partition of unity (PU) function with the enrichment function

$$\boldsymbol{u}(\boldsymbol{x})_E = \sum_{J \in N_E} \sum_{\alpha=1}^{n} N_J^{\star}(\boldsymbol{x}) \boldsymbol{\Psi}^{\alpha}(\boldsymbol{x}) \boldsymbol{a}_J^{\alpha} \tag{6}$$

where $\boldsymbol{\Psi}^{\alpha}(\boldsymbol{x})$ are enrichment functions, $N_J^{\star}(\boldsymbol{x})$ are the PU functions, N_E is the set of nodes enriched by $\boldsymbol{\Psi}^{\alpha}(\boldsymbol{x})$ and $\boldsymbol{a}_J^{\alpha}$ are the unknowns associated with the node *J* and the enrichment function α.

Standard FE approximation fulfills the Kronecker property, which leads to desirable features, namely the computed unknowns q_J are directly the values of displacement at the node J, thus the imposing the Dirichlet boundary condition $u_D(\boldsymbol{x})$ is simple $q_J = u_D(\boldsymbol{x}_J)$. In the standard XFEM approximation, these conditions do not hold. That can be achieved by modifying the enrichment function $\boldsymbol{\Psi}(\boldsymbol{x})$ as

$$\boldsymbol{\Psi}_{Shift}(\boldsymbol{x}) = \boldsymbol{\Psi}(\boldsymbol{x}) - \boldsymbol{\Psi}(\boldsymbol{x}_J) \tag{7}$$

which shifts the basis functions to zero point at each enriched node. Now, the enrichment term is given by

$$\boldsymbol{u}(\boldsymbol{x})_E = \sum_{J \in N_E} \sum_{\alpha=1}^{n} N_J^{\star}(\boldsymbol{x}) \boldsymbol{\Psi}_{Shift}^{\alpha}(\boldsymbol{x}) \boldsymbol{a}_J^{\alpha} \tag{8}$$

which leads to the desired property $\boldsymbol{u}(\boldsymbol{x}_J)_E = 0$.

4. Blending elements

In elements where only some of their nodes are enriched, functions $N_J^{\star}(\boldsymbol{x})$ do not build a partition of unity, i.e. $\sum_{J \in N_E} N_J^{\star}(\boldsymbol{x}) \neq 1$. In consequence, it can lead to large errors in the blending elements and poor global convergence. In particular, when XFEM is applied to linear elastic fracture mechanics, the parasitic terms in the approximation space of blending elements yield a decrease in accuracy of computed stress intensity factors.

An improvement in results is observed when the transition layer of blending elements is eliminated, and at the same time, the shifted-basis approximation is applied. This improvement is clearly manifested in models with coarse meshes, even without any additional technique e.g. point wise matching, and even for the problems with a higher order enrichment function, like the near tip function in the crack analysis.

5. Numerical example

The proposed approach has been applied to compute stress intensity factors for a beam in the plane stress state, with an edge crack, as shown in Fig. 2, where geometrical and material data for the test are also presented.

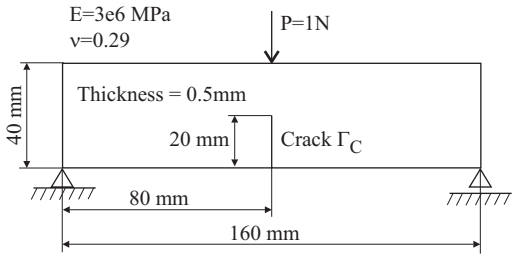

Figure 2: Beam with edge crack under concentrated force

Numerical calculations of SIF are carried out using the domain form of the interaction integral with the constant domain size radius $r_d = 10$ mm for each case. Table 1 shows the values of K_I for two different meshes used in discretization and the corresponding errors for both approaches.

Table 1: Numerical results

No. of elements	K_I XFEM	Error	XFEM shifted	Error
313	3.8069	13.64 %	3.2167	3.98 %
871	3.4674	3.50 %	3.2536	2.88 %

The error in stress intensity factors is calculated as

$$Error = \frac{\mid K_I^{Numerical} - K_I^{Theoretical} \mid}{K_I^{Theoretical}} \cdot 100\% \tag{9}$$

with the $K_I^{Theoretical} = 3.35$.

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