

Mechanics and thermodynamics of surface growth. Application to bone external remodeling

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Abstract

The surface growth of biological tissues is presently analyzed at the continuum scale of tissue elements, adopting the framework of the thermodynamics of surfaces. From a kinematic viewpoint, growth is assumed to occur in a moving referential configuration, considered as an open evolving domain exchanging mass, work, and energy with its environment. The growing surface is endowed with a superficial excess concentration of moles, which is ruled by an appropriate kinetic equation. The material surface forces for growth are evaluated versus a surface Eshelby stress, the curvature tensor of the growing surface, the gradient of the chemical energy of nutrients and the applied superficial force field. A system of coupled field equations is written for the superficial density of minerals, their concentration and the surface velocity of the growing surface. Application of the developed formalism to bone external remodeling highlights the interplay between transport phenomena and generation of surface mechanical forces. The model is able to describe both bone growth and resorption, according to the respective magnitude of the chemical and mechanical contributions to the material surface driving force for growth.

Keywords: tissue elements; surface growth; bone modeling and remodeling, surface diffusion; surface thermodynamics; surface Eshelby stress

1. Introduction

From a biological perspective, *surface growth* refers to mechanisms tied to accretion and deposition occurring mostly in hard tissues, and is active in the formation of teeth, seashells, horns, nails, or bones. A landmark in this field is [1], who describe the growth or atrophy of part of a biological body by the accretion or resorption of biological tissue lying on the surface of the body. Surface growth of biological tissues is a widespread situation, with may be classified as either fixed growth surface (e.g. nails and horns) or moving growing surface (e.g. seashells, antlers). Models for the kinematics of surface growth have been developed in [1], with a clear distinction between cases of fixed and moving growth surfaces, see [2] and the extensive list of references therein.

2. Kinematic viewpoint

We herewith adopt the point of view of the material body changing outside any physical time-dependent process, closely related to the kinematical concept of configurational mechanics, viz the physical deformed configuration is there kept fixed, while the inverse tangent mapping $\tilde{\mathbf{F}}^{-1}$ is a descriptor of modifications of the reference configuration, traducing deformations occurring in material space, see [3] and references therein. The material body consists of the set of material points which continuously fill a finite extent of the Euclidean space with matter, Truesdell and Noll (1965, 2004); in the case of growth, this set is in general not fixed, as new material points may be created or destroyed (the topology may change), inducing a variation of mass. The physical origin of this variation may be attributed to the exchange of chemical species (for instance nutrients) through the boundary of the material body, envisaged as an open system from a thermodynamical point of view.

As regards kinematics, the surface gradient $\tilde{\mathbf{F}}$ relating material points on the material boundary S_g of the growing body Ω_g

to their physical counterpart S_t (at the first order) is elaborated as the surface projection of \mathbf{F} onto the tangent plane to S_g (an alternative would be to consider a surface gradient independent of the volumetric gradient), $\tilde{\mathbf{F}} := \mathbf{F} \cdot \mathbf{P}$. Observe that contrary to volumetric growth, situations of compatible surface growth are likely to occur, due either to the absence of an existing residual strain field, or due to the fact that the newly added material may inherit the preexisting deformation field, a point of view already adopted by Frost in the early sixties. We essentially refer to such a situation in the present approach of surface growth.

3. Surface growth in the frame of surface thermodynamics

We consider a tissue element Ω_g with a portion of its boundary S_g undergoing surface growth surface and submitted to surface forces \mathbf{F}_s and line forces with densities O_t, P_n along the tangent and normal vectors $\mathbf{t}_g, \mathbf{n}_g$ to the surface contour δS_g (Fig. 1).

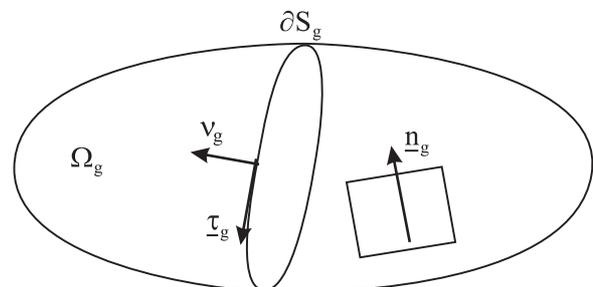


Figure 1: Tissue element under growth

The potential energy of the tissue element writes

$$V = \int_{\Omega_g} W_0 dX + \int_{S_g} \psi^S(\tilde{\mathbf{F}}, \mathbf{N}, n_k^\sigma) d\sigma_g + \int_{S_g} (\mu_k n_k^\sigma) d\sigma_g - \int_{S_g} \mathbf{F}_S \cdot \tilde{\mathbf{x}} d\sigma_g - \int_{\partial S_g} P_\tau \tilde{\mathbf{x}} \cdot \tau_g dl_g - \int_{\partial S_g} P_\nu \tilde{\mathbf{x}} \cdot \nu_g dl_g \quad (1)$$

considering an isothermal situation. The thermodynamics of surfaces is based on the setting up of *excess quantities*, (Linford, 1973), which give a physical significance to the growing surface. The surface energy contribution includes the surface energy density $\psi^S(\tilde{\mathbf{F}}, \mathbf{N}, n_k^\sigma; \mathbf{X}_S)$, depending upon $\tilde{\mathbf{F}}$, the normal to the growing surface \mathbf{N} to account for surface anisotropy, the excess concentration of moles n_k^σ , and possibly the surface position \mathbf{X}_S . The chemical potential m_k is dual to the variable n_k^σ and determines a chemical superficial energy $\mu_k n_k^\sigma$, reflecting the cell activity. The strain energy density in the bulk, $W_0(\mathbf{F}, \mathbf{X}_g)$, determines the nominal stress \mathbf{T} such that

$$W_0 = W_0(\mathbf{F}, \mathbf{X}_g) \rightarrow \mathbf{T} = \frac{\partial W_0(\mathbf{F}, \mathbf{X}_g)}{\partial \mathbf{F}} \quad (2)$$

The stationarity condition of the total potential energy, viz the condition $\delta V \equiv 0$, gives by localization onto S_g the necessary equilibrium condition:

$$\nabla_S \cdot \tilde{\Sigma} + \Sigma \cdot \mathbf{N} - \mathbf{P} \cdot \mathbf{K}^t \cdot \partial_N \psi^S + (\partial_{X_S} \psi^S)_{\text{expl}} + \mu_k \nabla_S n_k^\sigma + \mathbf{F}_S = \mathbf{0} \quad (3)$$

involving surface divergence of the surface energy momentum tensor

$$\tilde{\Sigma} := \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{Y}} - \psi^S \mathbf{U}_s \quad (4)$$

elaborated from the surface stress $\tilde{\mathbf{T}} := \partial_{\tilde{\mathbf{T}}} \psi^S$, $\partial_N \psi^S := \nabla_R \psi^{PS} \cdot \mathbf{N}$ a driving force (of vectorial nature) for rotations, the curvature tensor \mathbf{K} , the chemical contribution $\mu_k \nabla_S n_k^\sigma$, the surface density of forces \mathbf{F}_S (a datum of the problem), and the explicit derivative $(\partial_{X_S} \psi^S)_{\text{expl}}$ to account for surface inhomogeneities. The material surface driving force for surface growth expresses accordingly in terms as the sum of tangential and normal contributions, with only the normal driving forces given here

$$\Upsilon_{gN} := \mathbf{N} \cdot \Sigma \cdot \mathbf{N} + \tilde{\Sigma} : \mathbf{K} + \nabla_S \cdot (\tilde{\Sigma} \cdot \mathbf{N}) + \psi^S (\mathbf{N} \otimes \mathbf{N}) \cdot \nabla_S n_k^\sigma \quad (5)$$

The nutrients concentration n_k^σ satisfies the kinetic equation

$$\frac{\partial n^\sigma(t)}{\partial t} = -\tilde{\gamma} \rho_s(t) n^\sigma(t) \quad (6)$$

with $\tilde{\gamma}$ the surface absorption / resorption rate ; the diffusion terms have been neglected, as the diffusion of nutrients is generally much faster than the growth process. The surface mass balance equation in Eulerian format writes

$$\dot{\rho}_S + \rho_S \nabla_S \cdot \tilde{\mathbf{V}} = \Gamma^S \rho_S \quad (7)$$

The surface rate production of mass is given as

$$\mathbf{G}^S = \frac{dJ_{sg}}{dt} / J_{sg} = \mathbf{T}_r \cdot \left(\dot{\tilde{\mathbf{F}}} \cdot \tilde{\mathbf{F}}^{-1} \right) \quad (8)$$

4. Application to bone remodeling

From a mechanical point of view, bone is considered as a homogeneous continuum material composed mainly of hydroxyapatite, a type-I collagen, giving bone its structural rigidity. A one phase model is adopted, and the collagenous fraction will be discarded, as the mineral carries most of the strain energy in bone. In order to analyze the influence of mechanical stress on bone remodeling, we envisage a simple geometrical model of a long bone as a hollow homogeneous cylinder, endowed with a linear elastic isotropic behavior (the interstitial fluid phase in the bone is presently neglected), although an orthotropic behavior would be closer to reality. This situation is representative of the diaphyseal region of long bones, such as the human femur, which is modeled as a hollow cylinder. The elastic modulus is assumed to scale uniformly versus bone density following the experimentally based relation

$$E = E_{\text{max}}(t)^P, \quad E_{\text{max}} = 15 \text{ GPa} \quad (9)$$

The normal velocity of the growing surface (here the internal diaphyseal surface) is chosen as linearly related to the driving force evaluated from (5), calculated in the present context as

$$\tilde{\Upsilon}_{gN} = \frac{1}{r_i(t)} \left\{ \frac{1}{8} \Delta \mu n^\sigma(t) + \frac{A + 2B}{8(A + B)B} \sigma_{zz}^2 \right\} \quad (10)$$

Simulations show the decrease of the internal radius according to the applied axial stress σ_{zz} (Fig. 2), a situation of growth. For a stress level lower than a certain threshold, resorption would occur due to the dominant chemical energy contribution.

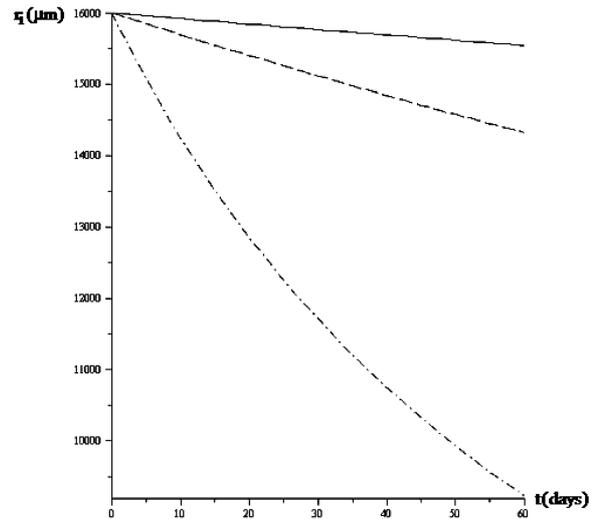


Figure 2: evolution vs. time of the internal radius of the diaphysis of the human femur. $\sigma_{zz} = 1$ MPa (thick line), $\sigma_{zz} = 2$ MPa (dashed line), $\sigma_{zz} = 5$ MPa (dash-dotted line)

References

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