Application of the Consistent R-Function Method to the solution of inverse problems

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Abstract

In the paper, the R-function method was modified and applied to solve inverse problems with an incomplete boundary condition.Using a simple example, we show good effectiveness of this analytical-numerical solution.

Keywords: boundary value problems, inverse problems, meshless methods

1. Introduction

The R-Function Method (RFM) can by classified as a meshless analytical-numerical method of solution of boundary value problems [4, 5]. In the paper, the method was modified assuming that the so-called *solution structure* is constructed in such a way that only the essential boundary conditions are satisfied. This simplified variant of the RFM, called Consistent R-Function Method (CRFM), was applied to the solution of inverse problems with incomplete boundary conditions.

2. Method of solution

Consider an elliptic problem defined by the *variational weak form*

$$G(v, u, \boldsymbol{\psi}) = 0, \quad \mathbf{x} \in \Omega \cup \partial\Omega_n \tag{1}$$

where $u(\mathbf{x})$ is an unknown function, $v(\mathbf{x})$ is an arbitrary function with compact support (so that $v(\mathbf{x}) = 0$ on $\partial\Omega$), and $\boldsymbol{\psi}$ is a vector of the natural boundary conditions

$$\boldsymbol{\psi} = \{\psi_1, \psi_2, ..., \psi_n\}, \quad \mathbf{x} \in \partial\Omega_n \tag{2}$$

The essential boundary conditions are collected in vector form

$$\boldsymbol{\varphi} = \{\varphi_1, \varphi_2, ..., \varphi_m\}, \quad \mathbf{x} \in \partial\Omega_m \tag{3}$$

with conditions $\partial\Omega = \partial\Omega_n \cup \partial\Omega_m$ and $\partial\Omega_n \cap \partial\Omega_m = 0$.

Let us assume, for example, that some of the essential boundary conditions are unknown

$$\begin{aligned} &\boldsymbol{\varphi} = \boldsymbol{\varphi}^k + \boldsymbol{\varphi}^u \\ &\boldsymbol{\varphi}^k = \{\varphi_1, \varphi_2, ..., \varphi_{m_1}\}^k, \ \boldsymbol{\varphi}^u = \{\varphi_1, \varphi_2, ..., \varphi_{m_2}\}^u \\ &m = m_1 + m_2 \end{aligned} \tag{4}$$

where φ_i^k and φ_i^u are a known and an unknown function, respectively. Instead, the values of $u_j, j = 1, 2, ..., p$ in Ω are known. The inverse problem defined above is solved using the Consistent R-Function Method and replacing the unknown essential boundary conditions by the *energetically adjoint* unknown natural boundary conditions.

Thus, the variational formulation (1) takes the form

$$G(v, u, \boldsymbol{\psi}, \boldsymbol{\psi}^u) = 0, \quad \mathbf{x} \in \overline{\Omega} \tag{5}$$

where $\overline{\Omega} = \Omega \cup \partial\Omega_n \cup \partial\Omega_{m_2}$ and

$$\boldsymbol{\psi}^u = \{\psi_1, \psi_2, ..., \psi_{m_2}\}^u, \quad \mathbf{x} \in \partial\Omega_{m_2} \tag{6}$$

with the essential boundary conditions

$$\boldsymbol{\varphi}^k = \{\varphi_1, \varphi_2, ..., \varphi_{m_1}\}^k, \quad \mathbf{x} \in \partial\Omega_{m_1} \tag{7}$$

The benefit of applying this method of the solution is obvious; the unknown essential boundary conditions do not appear in the solution structure and their approximation is not necessary. This conclusion can also be confirmed by comparing the method of solution with solution [1], where the classical R-function method is applied.

3. Example

Consider the following problem (see Fig. 1)

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = -4, \quad \mathbf{x} \in \Omega \in \mathbb{R}^2 \tag{8}$$

with the boundary conditions

$$\begin{aligned} &u(\mathbf{x}) = -2x_1 + 36, && \mathbf{x} \in \partial\Omega_1 \\ &u(\mathbf{x}) = \varphi^u, && \mathbf{x} \in \partial\Omega_2 \\ &u(\mathbf{x}) = 2x_2^2 - 16x_2 + 36, && \mathbf{x} \in \partial\Omega_3 \\ &\frac{\partial u}{\partial \mathbf{n}} = -2, && \mathbf{x} \in \partial\Omega_4 \end{aligned} \tag{9}$$

where $\mathbf{n}$ is the vector normal to $\partial\Omega_4$.
The problem has an exact solution

$$\varphi^u = 6x_1 - 2x_1^2 + 4 \tag{10}$$

It is also assumed that the three values of u in Ω are known and equal to

$$u_1(1, 4) = 2, \quad u_2(2, 5) = -2, \quad u_3(2, 4) = 2 \tag{11}$$

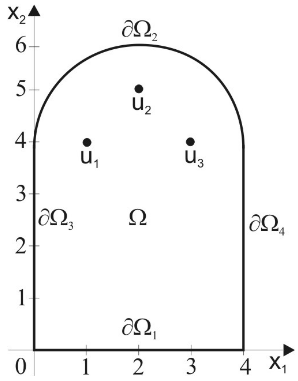

Figure 1: Solution domain

The variational form of Eq.(8) and the solution structure are given by

$$G = \int_{\Omega} \nabla u \cdot \nabla v d\Omega + 4\int_{\Omega} v d\Omega + 2\int_{\partial\Omega_4} v d\partial\Omega_4 - \int_{\partial\Omega_2} \frac{\partial u}{\partial \mathbf{n}} v d\partial\Omega_2 = 0 \tag{12}$$

$$u = \frac{(-2x_1 + 36)\omega_3 + (2x_2^2 - 16x_2 + 36)\omega_1}{\omega_1 + \omega_3} + \omega_1\omega_3\mathbf{\Phi c} = u_0 + \omega\mathbf{\Phi c} \tag{13}$$

where ω_1 and ω_3 are known functions that take zero values on the boundaries $\partial\omega_1$ and $\partial\omega_3$ and are positive in the interior of Ω

$$\omega_1 = x_2, \quad \omega_3 = x_1 \tag{14}$$

and u_0 satisfies the essential boundary conditions on $\partial\omega_1$ and $\partial\omega_3$, $\mathbf{\Phi}$ is the row matrix of the 21 Chebyshev polynomials and $\mathbf{c}$ is the vector of the unknown parameters. Substituting (13) into (12), we obtain a linear set of algebraic equations

$$\mathbf{Kc} = \mathbf{F} \tag{15}$$

where

$$\mathbf{K} = \int_{\Omega} \nabla(\omega\mathbf{\Phi}) \nabla (\omega\mathbf{\Phi})^T d\Omega - \int_{\partial\Omega_2} \frac{\partial(\omega\mathbf{\Phi})}{\partial \mathbf{n}} (\omega\mathbf{\Phi})^T d\partial\Omega_2$$

$$\mathbf{F} = -\int_{\Omega} \nabla u_0 \nabla (\omega\mathbf{\Phi})^T d\Omega - 4\int_{\Omega} (\omega\mathbf{\Phi})^T d\Omega + \int_{\partial\Omega_2} \frac{\partial u_0}{\partial \mathbf{n}} (\omega\mathbf{\Phi})^T d\partial\Omega_2 + 2\int_{\partial\Omega_4} (\omega\mathbf{\Phi})^T d\partial\Omega_4 \tag{16}$$

Equations (15) together with additional three equations, obtained by equating the solution structure (13) with the values of $u_j, j = 1, 2, 3$ (11), formed the final over-determined set of equations, which was solved with the help of the least-squares method.

The results of the computations are shown in Fig. 2. In this example, the number of measurement points and their arrangement in the vicinity of the unknown boundary condition did not have a significant influence on the final solution.

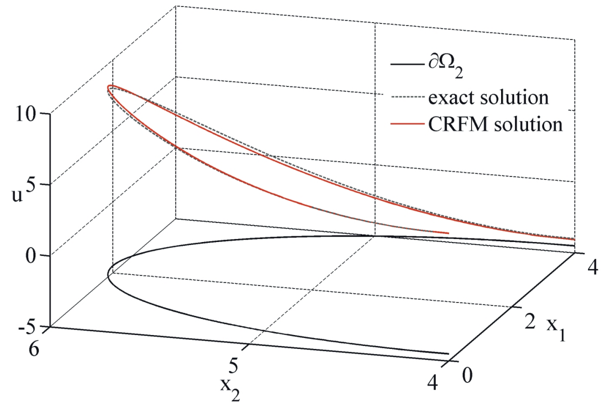

Figure 2: Comparison of the exact solution with the CRFM solution on boundary $\partial\Omega_2$

4. Conclusions

Convergence and uniqueness of the R-Function Method solution has been proved in [2, 3]. The consistent approach to the R-Function Method (CRFM) leads to the considerable simplification of the solution structure. The quality of the inverse problem solution using CRFM method depends on the choice of polynomials used in approximation and the proper location of measurement points. This should be establish through the tests.

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