# Optimization of thin plates made of the material with predefined eigenvalues of the elasticity tensor*

**Grzegorz Dzierżanowski and Tomasz Lewiński**
*Department of Structural Mechanics and Computer Aided Engineering, Warsaw University of Technology*
*al. Armii Ludowej 16, 00-637 Warsaw, Poland*
*e-mail: g.dzierzanowski@il.pw.edu.pl, t.lewinski@il.pw.edu.pl*

Abstract

The paper deals with the compliance minimization of a transversely homogeneous plate, subjected to the in-plane and transverse loadings acting simultaneously. The set of design variables includes the eigenstates of Hooke's tensor whose eigenvalues (or Kelvin moduli) fields are assumed fixed on the middle plane of the plate but no isoperimetric condition is imposed. The optimization task reduces to an equilibrium problem of an effective hyperelastic plate with strictly convex effective potential expressed in terms of strains. Theoretical considerations are illustrated by numerically calculated trajectories of the optimal eigenstate corresponding to the largest Kelvin modulus.

*Keywords: topology optimization, plates, anisotropy*

## 1. Introduction

Within the framework of thin, transversely homogeneous plates the bending stiffnesses $D^{\alpha\beta\gamma\delta}$ ($\alpha, \beta, \gamma, \delta$ run over $1, 2$) are determined by the in-plane stiffnesses $A^{\alpha\beta\gamma\delta}$ according to the formula $D^{\alpha\beta\gamma\delta} = (h^2/12)\, A^{\alpha\beta\gamma\delta}$, where $A^{\alpha\beta\gamma\delta} = h\, E^{\alpha\beta\gamma\delta}$, with $h$ denoting the plate thickness and $\boldsymbol{E}$ representing the elasticity tensor of the generalized plane stress. Thus the same tensor $\boldsymbol{A}$ determines both the in-plane and out-of-plane response of the plate. The topic of the present paper is the optimum design of plates subject to arbitrarily directed loads, and with tensors $\boldsymbol{A}$ belonging to a certain class $\mathcal{T}(\Omega)$, where the elements of $\mathcal{T}(\Omega)$ determine $\boldsymbol{A}$ at each point $x = (x_1, x_2)$ of the middle plane $\Omega$.

Irrespective of the type of loading the description of deformation of a transversely homogeneous plate splits up into the formulae defining the in-plane strains and bending curvatures. The in-plane displacements $(u_1(x), u_2(x))$ and the deflection $w(x)$ are found through independent boundary value problems. As a result of optimization these fields are coupled, since the target function usually involves both fields $\boldsymbol{u}$ and $w$. Among possible target functions a particular attention is payed to the compliance $C = f(\boldsymbol{u}, w)$, defined as the work of the loading on the unknown displacements (some authors divide $C$ by a norm of the loading, which is a formal change if the loading is deformation independent). It is worth pointing out that in case of compliance minimization the adjoint formulation coincides with the original one, which is a unique and remarkable feature of this problem. Moreover, the stress-based approach is possible, by the application of the Castigliano principle.

In the present paper we consider the minimum compliance problem of plates subject to an arbitrary loading. The design variables are the eigenstates of tensor $\boldsymbol{A}$ while its eigenvaules $\lambda_i(x)$, $i = 1, 2, 3$ are kept fixed in $\Omega$ and treated as parameters. These eigenvalues are called Kelvin moduli, see Ref. [2]. The fixing of Kelvin moduli replaces the isoperimetric condition which is always introduced into the Free Material Optimization (FMO) problems, see Ref. [1]. We prove that minimizing the compliance of a plate made of material with the constitutive tensor $\boldsymbol{A} \in \mathcal{T}(\Omega)$ is equivalent to the equilibrium problem of an effective hyperelastic plate with the potential $U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ locally determined by the maximization problem

$$U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2} \max_{\boldsymbol{A} \in \mathcal{T}(\Omega)} \left[ \boldsymbol{\varepsilon} \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}) + \boldsymbol{\kappa} \cdot (\boldsymbol{A}\boldsymbol{\kappa}) \right], \qquad (1)$$

where $\boldsymbol{\kappa} = (h/\sqrt{12})\boldsymbol{\varkappa}$; here $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ are the measures of in-plane and bending deformations. The symbol " $\cdot$ " stands for the scalar product in the sense $\boldsymbol{a} \cdot \boldsymbol{b} = a_{\alpha\beta}\, b_{\alpha\beta}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ denote symmetric tensors. By setting $\mathcal{T} = \mathcal{T}_\lambda$, where $\mathcal{T}_\lambda$ denotes the class of tensors $\boldsymbol{A}$ of given Kelvin moduli $\lambda_i$, $i = 1, 2, 3$, the potential $U$ can be determined explicitly, which is one of the aims of the present paper.

## 2. The optimum design problem

Consider a plate of constant thickness $h$ subject to the in--plane loading of intensity $\boldsymbol{p}(x) = (p_1(x), p_2(x))$ and to the transverse loading $q(x)$. Define the virtual work of these loadings on the test in-plane displacements $\boldsymbol{v}(x) = (v_1(x), v_2(x))$ and on the virtual transverse displacements $v(x)$ as the linear form

$$f(\boldsymbol{v}, v) = \int_\Omega (\boldsymbol{p} \cdot \boldsymbol{v} + q\, v)\, dx. \qquad (2)$$

Let $\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{\alpha\beta}(\boldsymbol{v}))$ stand for the symmetric part of the gradient $\nabla \boldsymbol{v}$, and set $\boldsymbol{\varkappa}(v) = (\varkappa_{\alpha\beta}(v))$, where $\varkappa_{\alpha\beta}(v) = -v_{,\alpha\beta}$ with $(\cdot)_{,\alpha}$ denoting partial differentiation $\partial/\partial x_\alpha$. The stress resultants $\boldsymbol{N} = (N^{\alpha\beta})$ and couple resultants $\boldsymbol{M} = (M^{\alpha\beta})$ are linked with the strains by linear equations: $N^{\alpha\beta} = A^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$, $M^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta}$, with $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{\varkappa} = \boldsymbol{\varkappa}(w)$; $\boldsymbol{u}, w$ denoting the unknown displacement fields. The plate is assumed transversely homogeneous, hence $\boldsymbol{A}$ and $\boldsymbol{D}$ depend only on $x \in \Omega$. At each fixed point $x \in \Omega$, tensor $\boldsymbol{A}$ admits the spectral decomposition, see Ref. [2],

$$\boldsymbol{A} = \lambda_1\, \boldsymbol{P}_1 + \lambda_2\, \boldsymbol{P}_2 + \lambda_3\, \boldsymbol{P}_3, \qquad (3)$$

with $\boldsymbol{P}_i = \boldsymbol{\omega}_i \otimes \boldsymbol{\omega}_i$, where $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ stand for the second rank tensors satisfying the orthogonality conditions $\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij}$.

*Acknowledgement: The paper was prepared within the Research Grant no N506 071338, financed by the Polish Ministry of Science and Higher Education, entitled: Topology Optimization of Engineering Structures. Simultaneous shaping and local material properties determination.

Tensors $\boldsymbol{P}_i$ have the properties of projectors and $\lambda_i$ are the numbers denoting the Kelvin moduli. According to Eqn (3) the tensor field $\boldsymbol{A}(x)$, $x \in \Omega$, is characterized by three symmetric tensor fields $\boldsymbol{\omega}_i(x)$ and three scalar fields $\lambda_i(x)$, $i = 1, 2, 3$. Consider now all admissible forms of $\boldsymbol{\omega}_i(x)$ (yet satisfying the orthogonality conditions) while $\lambda_i$ are kept fixed in $\Omega$. Such set of tensors fields $\boldsymbol{A}$ will be denoted by $\mathcal{T}_\lambda(\Omega)$.

Let us define the plate compliance $C = f(\boldsymbol{u}, w)$, where $(\boldsymbol{u}, w)$ solve the relevant equilibrium problem. The aim of the present paper is to find the plate of lowest compliance $C_0$ among the plates characterized by tensors $\boldsymbol{A}, \boldsymbol{D}$, such that $\boldsymbol{D} = (h^2/12)\boldsymbol{A}$ and $\boldsymbol{A} \in \mathcal{T}_\lambda(\Omega)$. Thus the compliance $C$ is viewed as a functional of $\boldsymbol{A}$, i.e. $C = C(\boldsymbol{A})$. Our aim is to find such distribution of the eigenstates $\boldsymbol{\omega}_i$ for which $C(\boldsymbol{A})$ assumes a minimal value

$$C_0 = \min_{\boldsymbol{A} \in \mathcal{T}_\lambda(\Omega)} C(\boldsymbol{A}), \quad \text{or} \quad C_0 = -2 \min_{(\boldsymbol{v}, v) \in \mathcal{V}} J_\lambda(\boldsymbol{v}, v) \tag{4}$$

where $\mathcal{V}$ represents the set of kinematically admissible displacements and

$$J_\lambda(\boldsymbol{v}, v) = \int_\Omega U_{\lambda(x)}(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\kappa}(v))\, dx - f(\boldsymbol{v}, v) \tag{5}$$

while

$$U_\lambda(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2} \max_{\boldsymbol{A} \in \mathcal{T}_\lambda} \left[\boldsymbol{\varepsilon} \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}) + \boldsymbol{\kappa} \cdot (\boldsymbol{A}\boldsymbol{\kappa})\right]. \tag{6}$$

The minimum compliance problem in Eqns (4) – (6) is equivalent to the equilibrium problem of an effective plate with hyperelastic properties. Indeed, the stationarity condition of $J_\lambda$ in Eqn (5) implies the virtual work equation in which the stress and couple resultants are linked with strains by

$$\boldsymbol{N} = \frac{\partial U_\lambda}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \qquad \boldsymbol{M} = \frac{h}{\sqrt{12}} \frac{\partial U_\lambda}{\partial \boldsymbol{\kappa}}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \tag{7}$$

and $\boldsymbol{\kappa} = (h/\sqrt{12})\boldsymbol{\varkappa}$.

The principal feature of the optimal design problem considered is the solvability of the local problem in Eqn (6). Its solution turns out to be

$$U_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \frac{1}{4}(\lambda_1 + \lambda_2)(||\boldsymbol{\varepsilon}||^2 + ||\boldsymbol{\kappa}||^2) + \\ + \frac{1}{4}(\lambda_1 - \lambda_2)[(||\boldsymbol{\varepsilon}||^2 - ||\boldsymbol{\kappa}||^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2]^{\frac{1}{2}} \tag{8}$$

where $\lambda_1 > \lambda_2$ and $\lambda_3$ does not affect the final result. Having found the strain fields $\boldsymbol{\varepsilon}, \boldsymbol{\kappa}$ one can recover the eigenstates $\boldsymbol{\omega}_i$ but this procedure is omitted here.

## 3. Examples of optimal design

For the simplicity of calculations we assume that the plate is made of the material such that $\lambda_1 > \lambda_2 = \lambda_3$. In this case, optimal tensor $\boldsymbol{A}$ is orthotropic and admits the form

$$\boldsymbol{A} = (\lambda_1 - \lambda_2)\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \lambda_2 \boldsymbol{I}_4, \tag{9}$$

where $\boldsymbol{I}_4$ denotes the unit tensor in the space of Hooke's tensors. It is worth pointing out that $\boldsymbol{A}$ becomes isotropic if and only if $\sqrt{2}\,\boldsymbol{\omega}_1 = \boldsymbol{I}_2$, where $\boldsymbol{I}_2$ is a unit tensor in the space of the second order symmetric tensors.

Let us consider a rectangular plate, see Fig. 1, whose in-plane displacements $u_1 = u_2 = 0$ along $A - B$ and the transversal displacement $w = 0$ along all the boundary of $\Omega$. The plate is subjected to the in-plane tractions along $C - D$ whose resultant equals $P$. The transversal load $q$ is uniform in $\Omega$.

Shown in the following figures are the trajectories of $\boldsymbol{\omega}_1$, i.e. the eigenstate corresponding to the greatest Kelvin modulus $\lambda_1$ of the constitutive tensor $\boldsymbol{A}$, optimally oriented against given deformation fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ whose trajectories are shown in Figs. 2, 3 respectively.

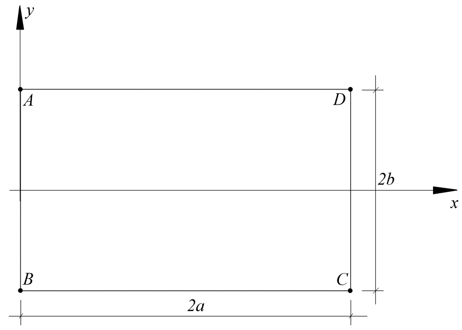


Figure 1: Middle plane $\Omega$ of a plate.

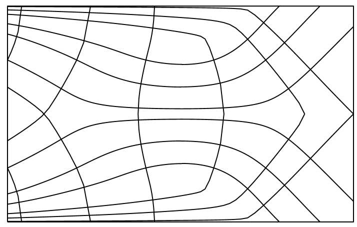

Figure 2: Trajectories of $\boldsymbol{\omega}_1(x, y)$.

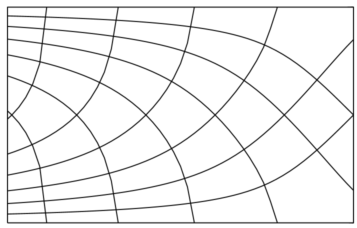

Figure 3: Trajectories of $\boldsymbol{\varepsilon}(x, y)$.

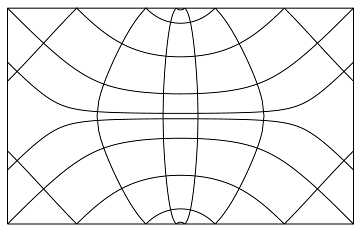

Figure 4: Trajectories of $\boldsymbol{\kappa}(x, y)$.

It turns out that the results of the present research can be understood as the generalization of the solutions to the constitutive tensor optimal orientation problems formulated for plates subjected to the pure in-plane or bending loadings. It is a matter of straightforward calculations to prove that optimal eigenstate $\boldsymbol{\omega}_1$ are in these cases given by $\boldsymbol{\omega}_1 = \boldsymbol{\varepsilon}/||\boldsymbol{\varepsilon}||$ and $\boldsymbol{\omega}_1 = \boldsymbol{\kappa}/||\boldsymbol{\kappa}||$ respectively.

## References

[1] Bendsøe, M.P., Guedes, J.M., Haber, R.B., Pedersen P., Taylor, J.E., An analytical model to predict optimal material properties in the context of optimal structural design, *ASME J. Appl. Mech.*, 61, pp. 930–937, 1994.

[2] Rychlewski, J., On thermoelastic constants, *Arch. Mech.*, 36, pp. 77–95, 1984.