Non-local approach in modelling of granular flow by the material point method

Zdzisław Więckowski[1] **and Iwona Kowalska–Kubsik**[2]

[1]*Department of Mechanics of Materials, Technical University of Łódź*
al. Politechniki 6, 90-924 Łódź, Poland
e-mail: zwi@p.lodz.pl

[2]*Department of Mechanics of Materials, Technical University of Łódź*
al. Politechniki 6, 90-924 Łódź, Poland
e-mail: Iwona.Kowalska-Kubsik@p.lodz.pl

Abstract

Large strain, granular flow problems are analysed in the paper. The mechanical behaviour of the granular material is described by the use of a non-local elastic–viscoplastic constitutive model. As a non-local variable, the rate of equivalent plastic strain is chosen. The large strain problem is solved by the use of the material point method. The results of the computations are shown in the case of two-dimensional examples of granular flow in a silo.

Keywords: meshless methods, finite element methods, dynamics, plasticity, granular flow

1. Introduction

Granular flow processes are characterized by appearing strain localisation zones in the form of shear bands. In the case of the use of classical constitutive models which do not satisfy the condition of material stability, $d_{ij}\,\dot\sigma_{ij} > 0$ (where d_{ij} denotes the rate-of-deformation tensor, σ_{ij} the Cauchy stress tensor, and the dot indicates the time derivative, e.g. [3]), the numerical solution is mesh dependent which means that the thickness of a shear band is determined by the size of elements of the computational mesh. In the dynamic case, the problem of mesh dependence can be healed, to some extent, by the use of the viscoplastic regularization of the constitutive relations [5]. This approach is not effective when the dynamic process is slow. On the other hand, it is not easy to find a direct relation between the viscosity parameter and the internal length characterizing the material. One of the easiest ways to get a proper solution of the considered problem is the use of a non-local constitutive model, e.g. [4]. An elastic–viscoplastic material model with the non-local effective plastic strain rate is employed in the paper.

A non-local constitutive model has been applied in the analysis of silo flow problems in [7] where an arbitrary Lagrangian–Eulerian finite element procedure has been utilised with the Euclidean norm of the rate-of-deformation tensor as a non-local variable within the hypoplastic constitutive model. Rather simple geometry of silos has been considered in this work, namely an individual converging hopper for which moderately distorted flow is observed.

2. Problem description

A dynamic, large strain problem of granular flow is analysed. The mechanical behaviour of the granular material is described by a non-local elastic–viscoplastic constitutive model. The Drucker–Prager yield condition and a non-associative flow rule implying the plastic incompressibility of the material are applied in the constitutive model. Let f denote the yield function $f(\sigma_{ij}) = q - mp$, where $m = 18\sin\varphi/(9-\sin^2\varphi)$ is a function of the angle of internal friction, φ, p and q are invariants of the stress tensor, $p = -\frac{1}{3}\,\sigma_{ii}$, $q = \sqrt{\frac{3}{2}\,s_{ij}s_{ij}}$, where $s_{ij} = \sigma_{ij} + p\,\delta_{ij}$ denotes the deviatoric part of the stress tensor. The constitutive relations for the elastic–viscoplastic model are as follows:

$$\dot p = K d_{kk}, \qquad e_{ij} = \frac{1}{2\,G}\,\overset{\triangledown}{s}_{ij} + \gamma\,\langle\Phi(f)\rangle\,\frac{\partial g}{\partial s_{ij}}, \tag{1}$$

where g denotes the plastic potential defined by the relation $g = q$. The following notation is used above: $d_{ij} = \frac{1}{2}\,(v_{i,j} + v_{j,i})$ is the rate-of-deformation tensor, $e_{ij} = d_{ij} - \frac{1}{3}\,d_{kk}\,\delta_{ij}$ is its deviatoric part, split into the hypo-elastic (e^{e}_{ij}) and viscoplastic (e^{vp}_{ij}) parts in the second equation of (1), $\overset{\triangledown}{\sigma}_{ij} = \dot\sigma_{ij} - \sigma_{ik}\,\omega_{kj} - \sigma_{jk}\,\omega_{ki}$ is the Zaremba–Jaumann rate of the stress tensor, $\omega_{ij} = \frac{1}{2}\,(v_{j,i} - v_{i,j})$ the spin, K and G are the bulk and shear moduli, respectively. Symbol γ in Equation (1) denotes the viscosity parameter, while the function defining the law of plastic flow has the form:

$$\langle\Phi(f)\rangle = \begin{cases}\Phi(f) & \text{if } f > 0\\ 0 & \text{if } f \le 0\end{cases}, \qquad \Phi\left(f(\sigma_{ij})\right) = \left(\frac{q - m\,p}{m\,p}\right)^N,$$

where $N > 0$.

As a non-local variable introduced into the constitutive model, the spatially weighted average of the equivalent viscoplastic strain rate is considered

$$\tilde e^{\mathrm{vp}} = \frac{\int_\Omega w(\boldsymbol{x},\boldsymbol{\xi})\,e^{\mathrm{vp}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}}{\int_\Omega w(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}} \quad \text{with} \quad e^{\mathrm{vp}} = \left(\frac{3}{2}e^{\mathrm{vp}}_{ij}e^{\mathrm{vp}}_{ij}\right)^{\frac{1}{2}} \tag{2}$$

where $w(\boldsymbol{x},\boldsymbol{\xi})$ denotes the weight function positive defined on a bounded support chosen in the form:

$$w = \begin{cases}\bar w & \text{if } s \le 1\\ 0 & \text{if } s > 1\end{cases}, \text{ where } s = \frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|}{r}, \tag{3}$$

symbol $\|\cdot\|$ denotes the Euclidean vector norm and parameter r is the radius of circular (spherical in 3D case) support of function w related to the intrinsic length of the material. Three forms of function $\bar w$ have been implemented: polynomial of the 3rd degree $\bar w = 1 + s^2(2s-3)$, constant function $\bar w = 1$, and Gaussian function $\bar w = \exp(-(2.5\,s)^2)$.

3. Material point solution

The problem is solved by the use of the material point method (MPM). The starting point in formulation of the final equations system of the method is the equation of virtual work:

$$\int_{\Omega} \varrho\,(a_i v_i + \frac{1}{\varrho}\sigma_{ij} v_{i,j})\,\mathrm{d}x = \int_{\Omega} \varrho\, b_i v_i\,\mathrm{d}x + \int_{\Gamma_\sigma} t_i v_i\,\mathrm{d}s + \int_{\Gamma_c} \sigma_{ij} n_j v_i\,\mathrm{d}s \quad \forall \boldsymbol{v} \in V_0,$$

where V_0 denotes the space of kinematically admissible fields of displacements. The three right hand side terms in the above equation are related to the volumetric, surface and contact forces. In MPM, the motion of material points defined independently of the computational mesh is traced by means of interpolation functions defined on the mesh. After use of the standard finite element interpolation procedure, the equation of virtual work leads to the following system of equations for vector of nodal accelerations $\mathbf{a}$:

$$\mathbf{M}\,\mathbf{a} = \mathbf{F} + \mathbf{F}_c - \mathbf{R}, \tag{4}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{F}$, $\mathbf{F}_c$ and $\mathbf{R}$ are the vectors of external, contact and internal nodal forces, respectively. The details can be found in [6]. The system of equations (4) is solved incrementally by means of an explicit Euler algorithm.

For each time increment, an iteration loop is performed. Firstly, the system of equations (4) and the stresses and the increment of the equivalent viscoplastic strain, Δe^{vp}, for each material point is calculated by the use of the return mapping procedure; next, an increment of the non-local variable (2) is calculated for each material point from the following formula:

$$\Delta\tilde{e}_i^{\mathrm{vp}} = \frac{\sum_j s_{ij}\Delta e_j^{\mathrm{vp}} V_j}{\sum_j s_{ij} V_j} \quad \text{where } s_{ij} = \frac{\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|}{r}, \tag{5}$$

index i is related to the point for which the non-local variable is calculated while index j denotes the numbers of points neighbouring point i including this point itself, V_j denotes the volume of the body sub-region represented by point j. Finally, new values of stresses are calculated after putting the values of the non-local variable to the constitutive relations, and the vector of nodal internal forces $\mathbf{R}$ is evaluated. The loop is repeated until the difference between the vectors of right-hand side of system (4) calculated in two consecutive iterations becomes sufficiently small.

4. Example

The plane flow during the process of discharging a silo having the rectangular cross-section is considered. The cross-section dimensions are: height 30 cm, width 20 cm. The outlet located symmetrically has the width of 10 cm. The following material data have been used in the computations: mass density $\varrho = 1500$ kg/m^3, Young's modulus $E = 1{\cdot}10^6$ Pa, Poisson's ratio $\nu = 0.3$, the angle of internal friction, $\varphi = 30^\circ$, the viscosity parameter, $\gamma = 1{\cdot}10^3$ s^{-1}, the exponent in viscosity law, $N = 1$. Calculations have been made assuming the weight function in the non-local law (2)–(3) in the form of the polynomial of the 3rd degree with various values of parameter r defining the radius of the function support. The results obtained for the value $r = 0.005$ m are shown in Fig. 1 in the form of maps of the non-local variable. The similar values of shear bands thickness are observed for meshes having elements of different sizes. Calculations performed with the use of the constant and Gaussian weight functions show similar results to that revealed in Fig. 1 (also in the case of flow pattern and flow rate), however, the maximum values of the effective plastic strain rate are slightly smaller in the case of these two functions.

The influence of parameter r on the flow rate of granular material has been also investigated. The smaller values of flow rate has been observed in the case of larger parameter r related to the internal length parameter and the size of material grains. A slower flow rate is observed in experiments in the case of larger grain diameter, e.g. [1].

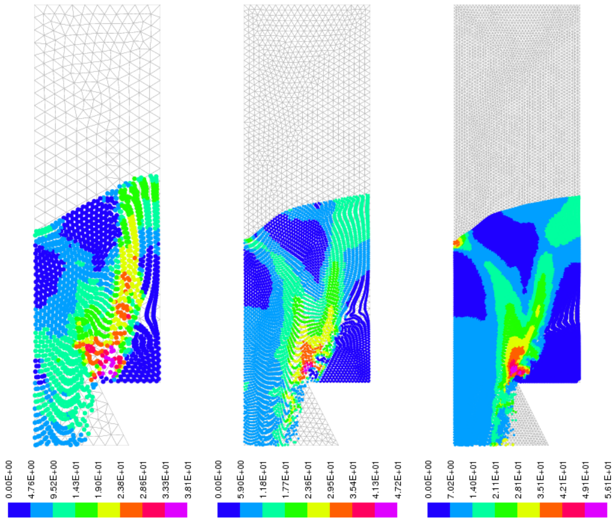

Figure 1: Effective rate of plastic strain (non-local variable), shown for one phase of discharging process (time = 0.25 s). The results are obtained for 3 meshes with different element sizes: 0.01, 0.005 and 0.0025 m

Comparison of the present results with those obtained by the use of the viscoplastic regularization of the constitutive relations [5] shows rather small differences. The flow rate and flow pattern are similar to each other when obtained by the two approaches. The results are in good agreement when the flow pattern and flow rate are compared with experiments, e.g. [1, 2]. The non-local approach, although it is much more time consuming, has a remarkable advantage comparing to the viscoplastic regularization technique as it can be used also in analyses of slow dynamic and quasi-static processes.

References

[1] Beverloo, W.A., Leniger, H.A. and van de Velde, J., The flow of granular solids through orifices, *Chem. Eng. Sci.*, 15, pp. 260–269, 1961.

[2] Coetzee, C.J., Els, D.N.J., Calibration of discrete element parameters and the modelling of silo discharge and bucket filling, *Computers and Electronics in Agriculture*, 65, pp. 198–212, 2009.

[3] Hill, R., A general theory of uniqueness and stability in elasto–plastic solids, *Journal of Mechanics and Physics of Solids*, 6, pp. 236–249, 1958.

[4] Pijaudier–Cabot, G. and Bažant, Z.P., Nonlocal damage theory, *Journal of Mechanical Engineering*, 113, pp. 1512–1533, 1987.

[5] Więckowski, Z., The material point method in the analysis of the problem of shear bounds formation, *Computational Fluid and Solid Mechanics*, Bathe K.J. Ed., Elsevier, Cambridge, MA, pp. 2173–2177, 2003.

[6] Więckowski, Z., The material point method in large strain engineering problems, *Comput. Methods Appl. Mech. Engrg.*, 193, pp. 4417–4438, 2004.

[7] Wójcik, M. and Tejchman, J., Modeling of shear localization during confined granular flow in silos within non-local hypoplasticity, *Powder Technology*, 192, pp. 298–310, 2009.