A review of new fundamental principles in exact topology optimization

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Abstract

The history of exact structural topology optimization will be reviewed and the optimal layout theory of Prager and Rozvany discussed in greater detail. Then some applications of new basic principles of topology optimization will be presented.

Keywords: exact topology optimization, trusses, grillages, Michell's theory, symmetry principles

1. The origins of exact structural topology optimization

1.1. Michell's truss theory

The optimality criteria for least weight trusses with a stress constraint and a single load condition were derived in 1904 by Michell [1] who extended a least weight theorem derived by Maxwell in 1872 [2]. The latter was restricted to trusses with either all compression or all tension members. Michell gave several examples of least-weight trusses, including those for a single point load between supports, a point load and a circular support and a truss along a spherical surface.

Cox in 1958 and 1965 [3][4] applied Maxwell's and Michell's criteria to some simple layout problems, including three concurrent forces and three parallel forces. He also showed that Michell's trusses for a stress condition also minimize the compliance. This was extended to other design conditions (plastic collapse load, natural frequency, stationary creep) by Hegemier and Prager in 1969 [5]. In a series of papers in 1958, 1966 and 1968 [6][7][8] and in a book in 1973 [9] Hemp derived Michell trusses for further boundary and loading conditions. For a cantilever with a point load ASL Chan in1960 [10] and HSY Chan in 1963, 1964 and 1967 [11][12][13] derived solutions for various limited lengths. Hemp in 1974 [14] proposed an exact solution for a distributed load between supports and HSY Chan [15] corrected somewhat this solution in 1975.

1.2. Optimal grillage theory

Prior to the development of the grillage-theory, some work was done on the mathematically analogous problem of plastically designed reinforced concrete plates. Wood derived in 1961 [16] the optimal reinforcement for simply supported and clamped circular plates and Rozvany in 1966 [17] for simply supported square plates (by a purely statical method). However, Morley (also in 1966) [18] derived powerful optimality criteria for the same problem and obtained the same reinforcement layout for simply supported square plates. He also extended this to rectangular plates. Melchers in 1972 and 1973 [19][20] applied Morley's optimality conditions to square and rectangular clamped domains, simply supported triangular boundaries, combinations of clamped and simply supported boundaries and reentrant corners.

Rozvany in 1972 [21] extended the above results to plastically and elastically designed grillages of given depth, and developed general rules for constructing systematically optimal grillage layouts for grillages with straight and curved supporting lines. These rules were used by Rozvany in 1972 [22] for deriving optimal grillage layouts for square and rectangular grillages with all possible combinations of simply supported, clamped and free boundaries. This was followed by a number of papers on optimal grillage topologies, which were summarized in review articles by Rozvany and Hill in 1976 [23] and Prager and Rozvany in 1977 [24]. The latest papers on grillage topology were by Rozvany and Liebermann in 1994 [25] on domains with some free (unsupported) edges, and by Rozvany in 1994 [26] on optimal support location and in 1997 [27] on partially downward and partially upward loading.

The remarkable feature of grillage topology optimization is that exact analytical solutions are available for almost all possible boundary and loading conditions and even a computer algorithm was developed for deriving analytically exact optimal grillage topologies (e.g. by Hill and Rozvany in 1985 [28]). Grillage layouts are more realistic than truss layouts, because buckling is not taken into consideration for the latter. Dense grillages, on the other hand, are rather stable structures.

1.3. Optimal layout theory (Prager and Rozvany 1977 [29], also in Rozvany's 1976 book [30])

This theory was created originally for grid-like structures with a low volume fraction, such as trusses, grillages, shell grids or dense systems of intersecting shells. However, it can be used for structures of higher volume fraction, as was shown by Rozvany, Olhoff, Bendsoe et al in 1985 and 1987 [31][32], who used it for perforated plates. Optimal layout theory has the following basic concepts.

The *ground structure* consists of all potential members, out of which the optimal ones are to be selected. In exact topology optimization the number of potential members is infinite, because they are of infinitesimal length at all points and in all directions within part of the space termed *feasible domain*.

The *specific cost* is the volume, weight or cost in a financial sense within a unit length, area or volume. For example, the specific cost of a plane truss at a point may be the material volume per a unit area.

The *optimality criteria* that follow are necessary and sufficient for convex problem and necessary for non-convex problems.

We have derived layout optimality criteria for all sorts of design constraints and multiple load conditions but for simplicity and clarity here we consider only stress constraints and a single load condition.

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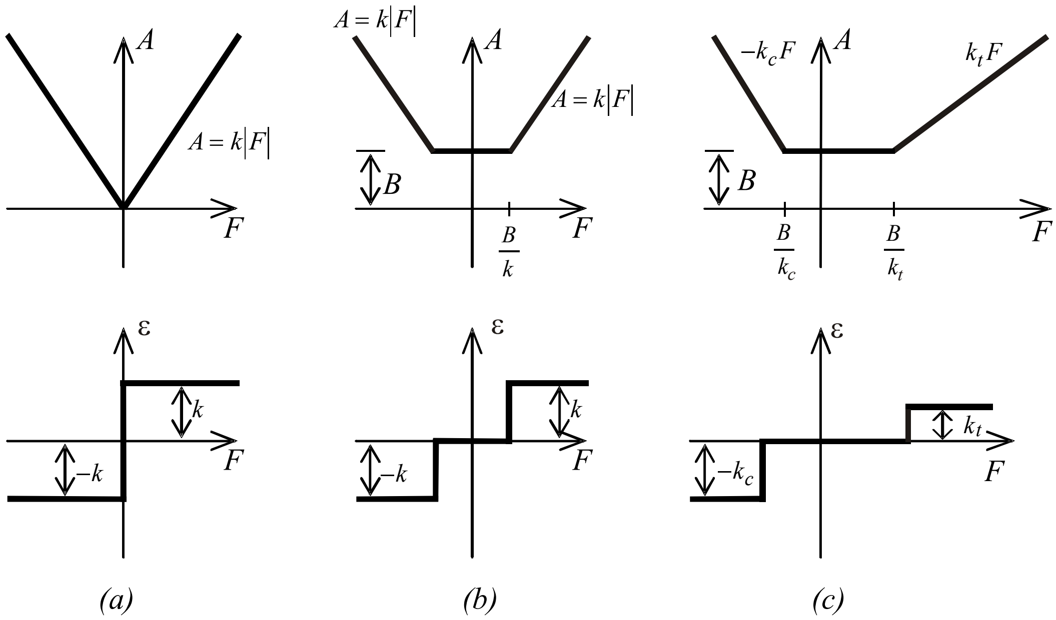

Figure 1: Specific cost functions and the corresponding adjoint strains for trusses with one load condition and a stress constraint: (a) Michell trusses, (a) and (b) equal permissible stress in tension and compression, (b) and (c) prescribed non-zero minimum cross sectional area, (c) different permissible stress in tension and compression

For sign-independent stress-based design of trusses and grillages of given depth, for example, the 'specific cost functions' (representing in this text cross sectional areas A are

$$A=k|F| \text{ and } A=k|M|, \tag{1}$$

where k is a constant, F is a member force, and M is a bending moment.

For trusses, this specific cost function is shown in Fig. 1a, top. For optimality of a layout we must calculate the actual stresses or stress resultants in all members. In some layout problems, we must also calculate the strains (termed 'real' strains).

In addition, we must find a so-called adjoint strain field, which is defined not only along members of the optimal layout, but over the entire feasible domain. Members of an optimal layout are sometimes called 'non-vanishing members', or 'optimal members' or 'members with non-zero cross-sectional area' in the literature.

The adjoint strain field is a fictitious one in general, but it must satisfy (a) kinematic continuity conditions, (b) kinematic support conditions, and (c) optimality conditions.

For the particular case considered here (stress constraint, one load condition), the optimality condition states that the adjoint strains must equal the subgradient of the specific cost function with respect to the stress resultant.

Then, e.g. for trusses (see the specific cost function in (1) above), the optimality conditions reduce to those of Michell (1904) [1]

$$\bar{\varepsilon} = k \operatorname{sgn} F \text{ (for } F \neq 0\text{)}, \quad |\bar{\varepsilon}| \le k \text{ (for } F = 0\text{)} \tag{2}$$

where $\bar{\varepsilon}$ is the adjoint strain.

For trusses, the optimal adjoint strains are indicated in Fig. 1a, bottom. Note that for a zero cross sectional area (zero force) the optimal adjoint strain is non-unique.

For Michell trusses, the above conditions are necessary and sufficient for optimality. These problems being 'self-adjoint', the adjoint strains and real strains are linearly proportional.

In Figs. 1b and c, we also give the specific cost functions and optimal adjoint strains for trusses with a prescribed minimum cross sectional area (B). In Fig. 1c the permissible stress is different in tension and compression.

For Michell trusses the adjoint strain field may consist of the following types of regions:

T-region with a tensile and a compression member at right angles, $\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = k$,

S-region with members having forces of the same sign in any direction, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2$, $|\varepsilon_i| = k$ $(i=1,2)$,

R-regions with only one member at any point, $|\bar{\varepsilon}_1| = k$, $|\bar{\varepsilon}_2| \le k$,

O-region with no members $|\bar{\varepsilon}_1| \le k$, $|\bar{\varepsilon}_2| \le k$,

where subscripts 1 and 2 indicate to principal strains. Depending on the sign of the forces, **S** and **R** regions may be further subdivided into $\mathbf{S}^+, \mathbf{S}^-, \mathbf{R}^+$ or $\mathbf{R}^-$ regions.

The optimal topology cannot be derived systematically, except for certain classes of problems, for which some optimal topologies are already known. For this reason the optimal topology of the regions must first be guessed ('dreamt up'), and then it can be checked if it can satisfy the above optimality criteria. At the same time, the optimal geometry of the optimal solution can be determined (see Fig. 2).

Alternatively, a good estimate of the region topology can be obtained by numerical solutions, and then the exact geometry can be calculated from the optimality conditions. Earlier on (for example for the 'MBB-beam' in the paper by Lewinski et al. in 1994 [33]) the numerical solution was obtained by the SIMP method using a perforated plate. More recently, higher resolution numerical solutions by Sokol's 2011 method [34] are used as a starting point for an analytical solution,

However, numerical solutions do not give a clue for the adjoint strains in O-regions, because those regions do not contain members (see Section 3.1).

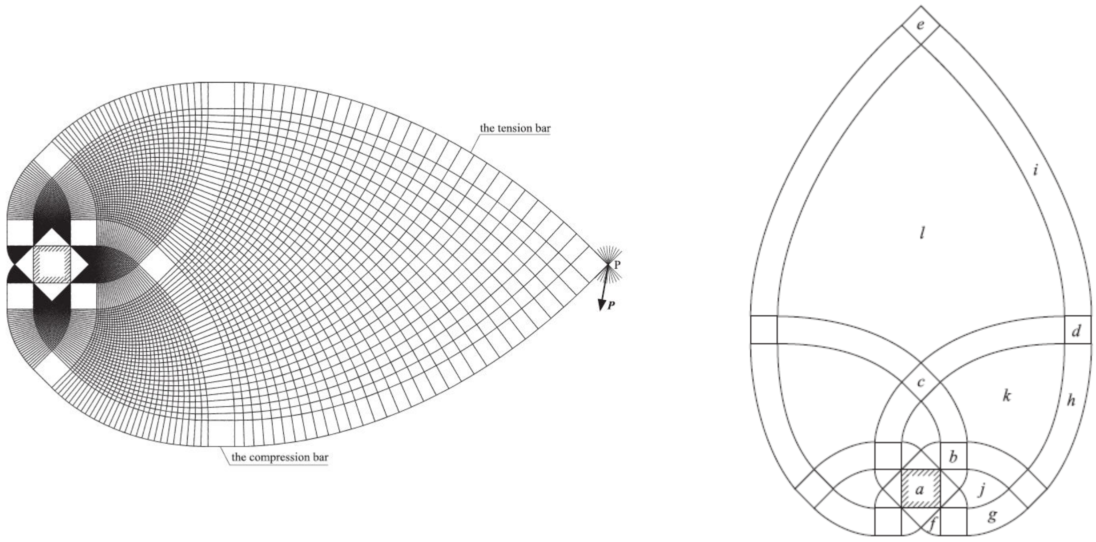

Figure 2: Left: exact analytical optimal Michell truss by Lewinski and Rozvany (2008) and Right: the correctly guessed region topology by Rozvany (1991)

2. More recent developments in truss topology optimization

Rozvany and Gollub (1990) [35] and Rozvany, Gollub and Zhou (1996) [36] derived Michell topologies for a large number of boundary conditions involving line supports. They even developed a computer program for determining the optimal topology analytically for any convex polygonal boundary.

In a paper Rozvany (1996) [37] (a) pointed out an error in Michell's (1904) derivation of his optimality criteria, (b) derived the correct optimality condition by three different methods for different permissible stresses in tension and compression, (c) stated the (limited) range of validity of Michell's original criteria, and (d) showed on a simple example significantly lower structural volume for the new criteria.

One should point out that Hemp (1973) [9] stated the above optimality conditions correctly, but has not indicated which examples of Michell are erroneous, nor given any example using the new criteria.

Lewinski et al. 1994 [38][33] derived the correct extended optimal topologies for a so-called Michell cantilever and the 'MBB-beam'. Both have been used as frequent benchmarks ever since.

Using the correct optimality criteria by Hemp (1973) [9] and Rozvany (1996) [37], Graczykowski and Lewinski T (2006/2007) [39], (2010) [40] derived a number of solutions for different permissible stresses in tension and compression.

Lewinski and Rozvany (2007/2008) [41][42][43] obtained the optimal Michell layout for various polygonal supports and for an L-shaped domain. The exact solution for a square shaped support and its earlier prediction are shown in Fig. 2.

Sokol and Lewinski (2010) [44] derived a comprehensive set of solutions for the three force problem.

3. New fundamental principles in exact structural topology optimization

Recently several new *fundamental principles* of exact structural topology optimization have been developed (Rozvany 2011 [45][46]), which can be used for identifying *basic properties* of these topologies, and also for deriving *new optimal topologies* on the basis of existing, known solutions.

The *domain augmentation principle* (Rozvany, 2011 [46]) states the conditions under which an optimal structural topology does not change if we modify the boundary line supports in such a way that (i) active supports along the old boundary are contained in the new boundary, and (ii) no point of the new boundary is contained in the interior of the old domain.

Example. In Fig. 1, the optimal Michell truss for the old boundary (triple line) was derived earlier (e. g. Rozvany and Gollub 1990, [35] p. 1030). For this optimal layout, the load must act above the line with the 2:1 slope (see Fig.3). Based on the above theorem, the same truss topology is optimal for the new boundary (dotted line in Fig. 3).

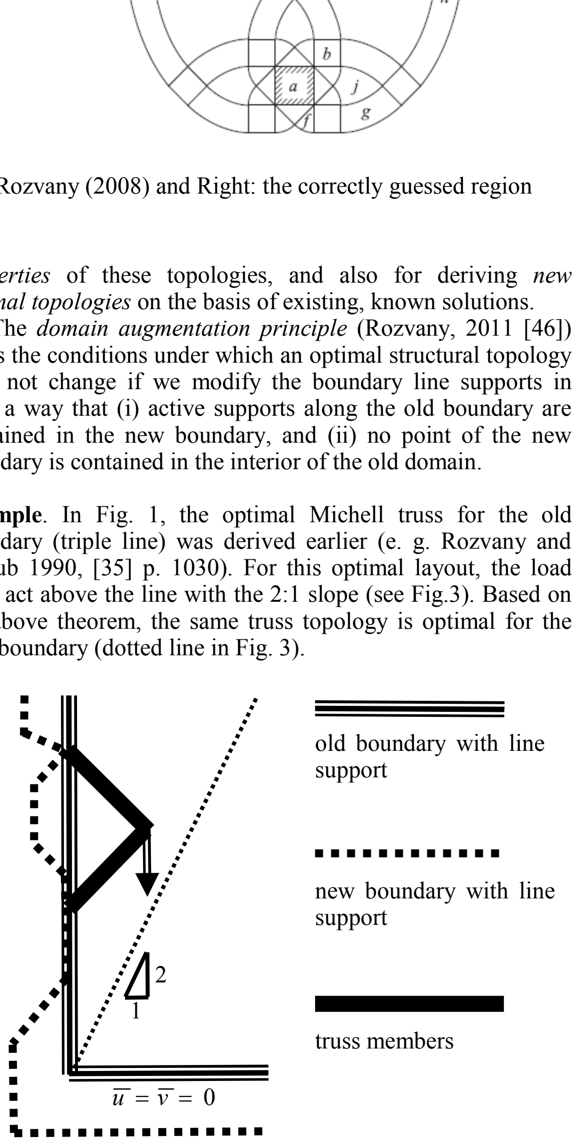

Figure 3: Example of domain augmentation

Similarly, the *domain reduction principle* states that optimal Michell topology remains the same, if we change the design domain in such a way that (i) all points of the new boundary are contained in the old domain, (ii) the original optimal Michell truss (including its active supports) is fully contained in the new domain.

Example. The optimal topology for two intersecting line supports in Fig. 4a can be calculated from simple formulae derived by Rozvany and Gollub (1990, [35] Fig. 15a). The direction of the point load is non-unique for this topology, it may vary within ±35°. By using the domain reduction theorem with the new boundary shown in broken lines, we obtain the optimal topology for two hinge supports (Fig. 4b). This solution is nor trivial, because for point loads closer to the vertical, the optimal topology becomes a 'Michell cantilever', with some fans and doubly curved Hencky nets in it (Lewinski et al 1994, [38]).

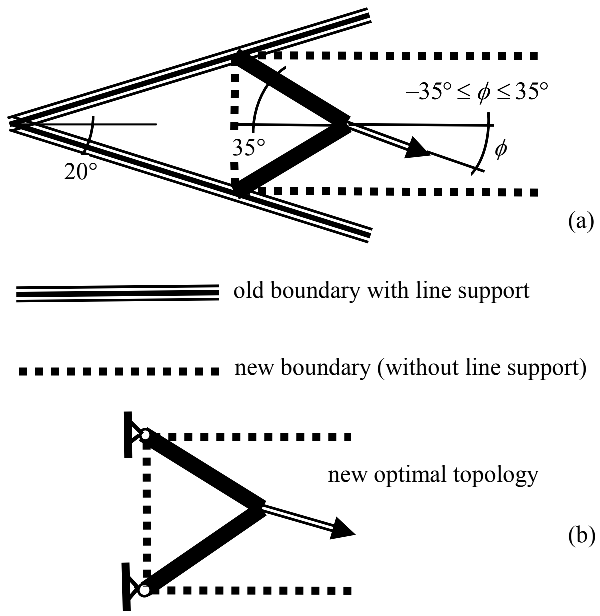

Figure 4: Example for the domain reduction principle

Symmetry principles investigate the conditions, under which the optimal topology (or at least one such topology) is symmetrical. Similarly, *skew-symmetry principles* deal with conditions for skew-symmetric topologies. *Non-uniqueness principles* show, in which cases there exists more than one optimal topology (Rozvany 2011 [45]).

Applications of the above principles for deriving new optimal topologies will be illustrated with a number of examples. These examples will include both least weight (Michell) trusses and optimal grillages.

Some of these new solutions have been verified numerically by Sokol (e.g. Sokol and Rozvany 2011, [48]), who has used up to two hundred million potential members in the ground structure and obtained excellent agreement with the analytical solution. It will be shown that for quite simple support conditions the optimal topology may take on several forms, depending on the load location, some of which are highly complicated.

3.1. The problem with O-regions

Most Michell truss researchers employ T-regions as a rule, because the theory of such regions is highly developed, and the mathematically similar slip-line theory of plasticity can be used for guidance.

The optimal adjoint strains in regions without members are difficult to determine, because numerical solutions do not give any clues for such regions, and the theory for O-regions has not been developed. Fig. 5 shows two Michell topologies using O-regions, but the one in Fig. 5b is only valid if the horizontal displacements are prevented at the loads (e. g. by rollers). The earlier mentioned symmetry theorems (Rozvany 2011 [45] had been employed for these problems. The problem of O–regions was also pointed out by Melchers in 2005 [49].

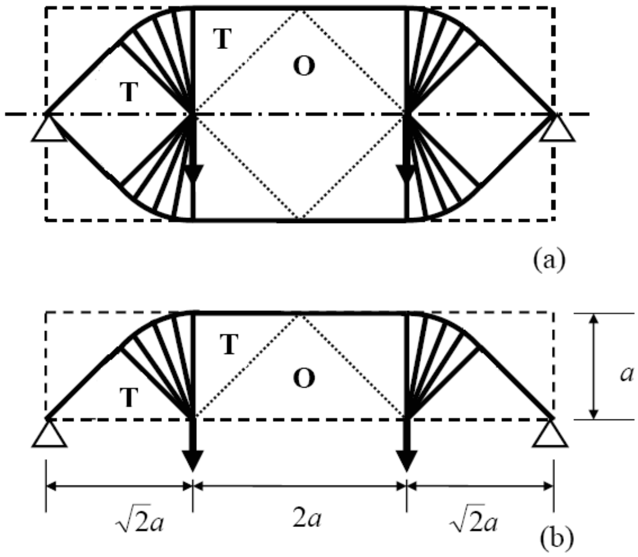

Figure 5: Examples of Michell topologies with O-regions

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