Free material design in compliance minimization of 3D elastic body

Sławomir Czarnecki*

Department of Structural Mechanics and Computer Aided Engineering, Warsaw University of Technology
al. Armii Ludowej 16, 00-637 Warsaw, Poland
e-mail: s.czarnecki@il.pw.edu.pl

Abstract

The paper deals with the minimum compliance problem formulated within the framework of 3D elasticity. Basing on spectral decomposition of Hooke's tensor, optimum distribution of the Kelvin moduli is found together with the optimum layouts of all components of the anisotropic elasticity tensor in 3D body. Moreover the stress trajectories for optimal anisotropic body and the layouts of two moduli defining the isotropic material closest (in appropriate meaning) to the optimal anisotropic material are shown.

Keywords: anisotropy, composites, elasticity, finite element methods, meshless methods, numerical analysis, optimization

1. Introduction

Majority of papers concerning the problem of maximization of the overall stiffness of a structure base on the relaxation by homogenization method (see e.g. [1]). The alternative method is FMO (Free Material Optimization) where the components of the Hooke elasticity anisotropic tensor are treated as design variables. The subject of the present paper is reformulation of this approach, similarly to that presented in [4] and using the Rychlewski spectral representation theorem (see e.g. in [2]). The Kelvin moduli are viewed as depending on a density function, as in SIMP method, while the elastic distributors are free of any restrictions, thus making it possible to lay out the Kelvin moduli optimally, according to the loading applied. The boundary value problem involved is solved by FEM or/and MFM (Mesh Free Method). Method of Moving Asymptotes (MMA) [3] is used as the optimizer for Kelvin moduli. Moreover, the stress trajectories are shown and compared with the optimal layouts of elastic distributors defining the anisotropic elasticity tensor. The properly dense set of points lying on the three appropriate families of spatial curves – stress trajectories – is created on the base of the simplest version of well known Runge-Kutta method of finding numerical solutions of the system of ordinary differential equations. The last group of the results, in this relative comprehensive visualization spectrum of various optimal solutions, is determined by the layouts of Young modulus and Poisson coefficient defining the isotropic material closest to the optimal anisotropic material in the sense of two metrics: conventional Frobenius and log-Euclidean distance (see e.g. [5]).

2. Formulation of an optimal design

Consider an elastic body Ω. The body is subject to a boundary loading $\mathbf{T}$ acting on a part Γ_1 of the boundary $\partial\Omega$ of a given 3D domain Ω, parameterized by Cartesian coordinates (x_1, x_2, x_3) with the orthonormal basis $\{\mathbf{e}_i\}_{i=1,2,3}$. The body is fixed along Γ_2 being a part of $\partial\Omega$, i.e. $\mathbf{u}\big|_{\Gamma_2} = \mathbf{0}$, where $\mathbf{u} = (u_1, u_2, u_3)$ are the unknown displacement fields referred to Ω – the response of the material on the boundary loading $\mathbf{T}$. Let $\mathbf{C}$ be the tensor of elastic moduli, i.e. $\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\varepsilon}$, where $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ (i, j=1,2,3) are respectively the symmetric stress tensor and symmetric strain tensor $\boldsymbol{\varepsilon} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ being the symmetric part of the displacement gradient $\nabla\mathbf{u} = (\partial u_i / \partial x_j)$. At each point $x \in \Omega$ tensor

$$\mathbf{C}(x) = \sum_{i,j,k,l=1}^{3} C_{ijkl}(x)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \sum_{K=1}^{6} \lambda_K(x)\,\mathbf{P}_K$$

is characterized by its eigenvalues $\lambda_K(x)$ called Kelvin moduli and the corresponding projection operators $\mathbf{P}_K = \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_K$ being tensorial products of so called eigenstates $\boldsymbol{\omega}_K(x)\,(K = 1,...,6)$, $\boldsymbol{\omega}_K(x)\cdot\boldsymbol{\omega}_L(x) = \delta_{KL}$. We shall assume that $\lambda_K(x) = \lambda_K^0\,\rho(x)^s\,(s > 0)$ where density field $\rho : \overline{\Omega} \to \mathbb{R}$ vary within the following given limits $\rho_{\min} \le \rho(x) \le \rho_{\max}$, $\int_\Omega \rho(x)dx \le V$, $0 < \rho_{\min} < \rho_{\max}$, $V > 0$, and $\lambda_1^0 \ge \lambda_2^0 \ge ... \ge \lambda_6^0 > 0$ are known. Let us consider the equilibrium problem of the body subject to the loading $\mathbf{T}$:

$$\mathbb{Z} = \min_{\text{kinematically admissible } \mathbf{v},\boldsymbol{\tau},\mathbf{e}} \left(\frac{1}{2}\int_\Omega \boldsymbol{\tau}\cdot\mathbf{e}\,dx - \int_{\Gamma_1} \mathbf{T}\cdot\mathbf{v}\,da \right) \text{ where } \mathbf{v}\big|_{\Gamma_2} = \mathbf{0},$$

$\boldsymbol{\tau} = \mathbf{C}\,\mathbf{e}$, $\mathbf{e} = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$. The minimal value $\mathbb{Z}$ of the above functional for the solution $\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ is equal to $-\frac{1}{2}\int_{\Gamma_1} \mathbf{T}\cdot\mathbf{u}\,da$. More convenient is to formulate the equilibrium problem equivalently as $\mathbb{C} = \min_{\text{statically admissible } \boldsymbol{\tau}} \left(\int_\Omega \boldsymbol{\tau}\cdot\mathbf{C}^{-1}\boldsymbol{\tau}\,dx \right)$ where $\mathbf{div}\,\boldsymbol{\tau} = \mathbf{0}, \boldsymbol{\tau}\big|_{\Gamma_1}\mathbf{n} = \mathbf{T}$ ($\mathbf{n}$ is normal to $\partial\Omega$) because the sum $\mathbb{Z} + \frac{1}{2}\mathbb{C}$ of the functionals for both solutions $\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ respectively, is equal to 0, so the optimal value $\mathbb{C}$ can be interpreted as the compliance of the body in the equilibrium state, i.e. $\mathbb{C} = \int_{\Gamma_1} \mathbf{T}\cdot\mathbf{u}\,da$. Let us notice that the optimal value $\mathbb{C}$ is dependent, via elasticity tensor $\mathbf{C}(x)$ or $\mathbf{C}^{-1}(x)$ of the density field $\rho = \rho(x)$ and eigenstates $\boldsymbol{\omega}_K = \boldsymbol{\omega}_K(x)$. So, it is possible additionally to minimize the $\mathbb{C} = \mathbb{C}(\rho, \boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_6)$ with respect to the density field ρ and eigenstates $\boldsymbol{\omega}_K$. Defining (in

* Acknowledgement. The paper was prepared within the Research Grant no N506 071338, financed by the Polish Ministry of Science and Higher Education, entitled: Topology Optimization of Engineering Structures. Simultaneous shaping and local material properties determination.

obvious way), the admissible set $\Re = \left\{ \rho : \overline{\Omega} \to \mathbb{R}; \int_{\Omega} \rho(x)\,dx \le V,\ \rho_{\min} \le \rho(x) \le \rho_{\max},\ x \in \overline{\Omega} \right\}$ of density fields ρ and the admissible set of eigenstates $\boldsymbol{\omega}_K$ as $\mathbb{Q} = \left\{ \boldsymbol{\omega} = (\boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_6) : \overline{\Omega} \to S^3 \times ... \times S^3;\ \boldsymbol{\omega}_K(x) \cdot \boldsymbol{\omega}_L(x) = \delta_{KL},\ \left\| \boldsymbol{\omega}_K(x) \right\| = 1,\ x \in \overline{\Omega} \right\}$ we can define our problem: find $\mathbb{C}_{\min} = \min_{(\rho,\boldsymbol{\omega}) \in \Re \times \mathbb{Q}} \ \min_{\text{statically admissible } \boldsymbol{\tau}} \left(\int_{\Omega} \boldsymbol{\tau} \cdot \mathbf{C}^{-1} \boldsymbol{\tau}\, dx \right)$. It is well known that $\mathbb{C}_{\min}$ can be found equivalently as the solution of the problem: find $\mathbb{C}^1_{\min} = 2 \min_{\rho \in \Re} \ \min_{\text{statically admissible } \boldsymbol{\tau}} \Psi(\rho, \boldsymbol{\tau})$, where $\mathbb{C}_{\min} = \mathbb{C}^1_{\min}$ and $\Psi(\rho, \boldsymbol{\tau}) = \frac{1}{2} \int_{\Omega} \frac{\left\| \boldsymbol{\tau} \right\|^2}{\lambda_1^0 \rho^s} dx$ is the functional of complementary energy for material characterized by the elasticity tensor $\mathbf{C} = \lambda_1^0 \rho^s\, \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1,\ ,\ \boldsymbol{\omega}_1 = \boldsymbol{\tau} / \left\| \boldsymbol{\tau} \right\|$. Consequently, the solution of the equilibrium problem of the body subject to the loading $\mathbf{T}$ and characterized by the optimal elasticity tensor $\mathbf{C}^1_{\min} = \lambda_1^0\, \rho^s_{\min} \boldsymbol{\omega}_1^{\min} \otimes \boldsymbol{\omega}_1^{\min}$, $\boldsymbol{\omega}_1^{\min} = \boldsymbol{\sigma} / \left\| \boldsymbol{\sigma} \right\|$, where $\rho = \rho_{\min} \in \Re$ and statically admissible stress field $\boldsymbol{\tau} = \boldsymbol{\sigma}$ are the optimal minimizers of functional $\Psi(\rho, \boldsymbol{\tau})$, is exactly equal to the solution of the equilibrium problem of the body subject to the loading $\mathbf{T}$ and characterized by the elasticity tensor $\mathbf{C}_{\min} = \lambda_1^0\, \rho^s_{\min} \boldsymbol{\omega}_1^{\min} \otimes \boldsymbol{\omega}_1^{\min} + \sum_{K=2}^{6} \lambda_K^0\, \rho^s_{\min} \boldsymbol{\omega}_K^{\min} \otimes \boldsymbol{\omega}_K^{\min}$, where $\boldsymbol{\omega}_{\min} = \left(\boldsymbol{\omega}_1^{\min}, \boldsymbol{\omega}_2^{\min}, ..., \boldsymbol{\omega}_6^{\min} \right) \in \mathbb{Q}$. So, as a rule, the full (but not unique) formula for the minimizer $\boldsymbol{\omega}_{\min}$ and consequently for the tensor $\mathbf{C}_{\min}$ is not revealed. In the paper, the optimal layouts of all components of the tensor $\mathbf{C}_{\min}$ in the base $\mathbf{w}_i \otimes \mathbf{w}_j \otimes \mathbf{w}_k \otimes \mathbf{w}_l$ defined by tensor product of three eigenvectors $\mathbf{w}_i = \mathbf{w}_i(x) \in \mathbb{R}^3$ (i=1,2,3) of an eigenstate $\boldsymbol{\omega}_1^{\min} = \boldsymbol{\omega}_1^{\min}(x)$, $x \in \Omega$ and additionally in the global base $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ are shown. Another important contribution to the designing of the optimal elastic material could be the comparison of compliances together with the visualization of the layouts of the isotropic elasticity tensor $\mathbf{C}^{\text{iso}}_{\min}$ (characterized by constants κ and μ or Young modulus E and Poisson coefficient ν) closest to a given and optimal anisotropic tensor $\mathbf{C}_{\min}$ (see e.g. [5]) in the sense of conventional Frobenius metric $\left\| \widehat{\mathbf{A}} - \widehat{\mathbf{B}} \right\|$ or the log-Euclidean distance $\left\| \text{Log}\,(\widehat{\mathbf{A}}) - \text{Log}\,(\widehat{\mathbf{B}}) \right\|$. In our case the $\widehat{\mathbf{C}} = \widehat{\mathbf{C}}(\mathbf{C})$ denotes the 6 × 6 matrix of components of the elasticity tensor $\mathbf{C}$ and $\left\| \widehat{\mathbf{C}} \right\| = \sqrt{\text{tr}\,(\widehat{\mathbf{C}}^T \widehat{\mathbf{C}})}$. The optimal layouts of the constants κ and μ or E and ν are of course not dependent on the choice of bases: $\mathbf{w}_i \otimes \mathbf{w}_j \otimes \mathbf{w}_k \otimes \mathbf{w}_l$ or $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$. Lastly, connected with the optimal stress solution, the trajectories of the tensor $\boldsymbol{\omega}_1^{\min} \sim \boldsymbol{\sigma}$ are shown. To this end, the three systems (for succeeding i = 1, 2, 3) $\frac{dr_j}{ds}(s) = w_{ij}(s, \mathbf{r}(s))$ $(j = 1,2,3)$ of ordinary differential equations (ODE) with the initial conditions $r_j(0) = x_j$ $(j = 1,2,3)$ are defined, where s is a natural parameter of the sought curve – trajectory $\mathbf{r}(s) = \sum_{j=1}^{3} r_j(s)\, \mathbf{e}_j$ and $\mathbf{w}_i(s, \mathbf{r}(s)) = \sum_{j=1}^{3} w_{ij}(s, \mathbf{r}(s))\, \mathbf{e}_j$ is the function defining the i-th eigenvector of the known stress tensor $\boldsymbol{\sigma}(s, \mathbf{r}(s))$ in spatial point $\mathbf{r}(s) \in \mathbb{R}^3$ lying on the trajectory and distant s from the beginning $\mathbf{x}$, cf Fig.1. Eigenvectors are always assumed to be a sorted list with respect to the three eigenvalues. The Runge-Kutta method was adopted to find the numerical solution of the above differential equations.

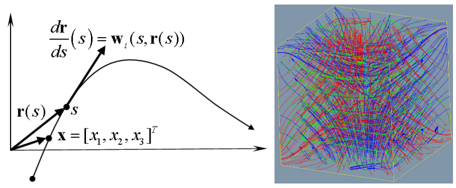

Figure 1: Trajectory determined by i-th eigenvector $\mathbf{w}_i$ of the stress tensor $\boldsymbol{\sigma}$ (on left) and example of the three families of stress-trajectories found numerically by the Runge-Kutta method for known analytical solution of an elastic cube loaded skew symmetrically (at the right-hand side)

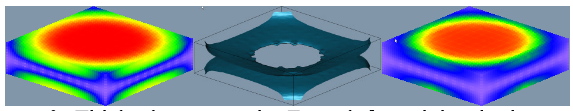

Figure 2: Thick plate example. From left to right, the layout of C_{2323} component of the optimal anisotropic elasticity tensor $\mathbf{C}$, with its one isovalue graph and optimal layout of the modulus κ of the equivalent isotropic plate

The approach proposed has to be supplemented with numerical recipes enabling to find the solution of the boundary problem of linear elastostatics, such as FEM or MFM. However, to find numerically the solution of the ODE, one has to know how relatively easily and quickly calculate all components of the eigenvectors of the stress tensor in any point $x \in \Omega$ of the body (as a rule, high precision determining the huge number of points is necessarily required in such methods). Shape functions in many variants of MFM are defined directly on real body (not on master element as in FEM), so MFM are in such calculations much more convenient to implement.

References

[1] Bendsøe, M.P., *Optimization of Structural Topology, Shape and Material*, Springer, Berlin, 1995.
[2] Rychlewski, J., On thermoelastic constants, *Arch. Mech.*, 36, pp. 77–95, 1984.
[3] Svanberg K., The method of moving asymptotes – a new method for structural optimization, *Int.J. Numer. Meth. Engin.*, 24, pp.359-373, 1988
[4] Turteltaub, S. and Washabaugh, P., Optimal distribution of material properties for an elastic continuum with structure-dependent body force, *International Journal of Solids and Structures*, 36, pp. 4587–4608, 1999.
[5] Norris, A, N., The isotropic material closest to a given anisotropic material, Journal of Mechanics of Materials and Structures, Vol. 1, No. 2, pp.223–23 8, 2006