Cahn-Hilliard system for microstructure evolution in viscoelastic solids. Asymptotic behaviour

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Abstract

In this paper the long-time behaviour of a unique regular solution to the Cahn-Hilliard system coupled with viscoelasticity is studied. The system arises as a model of phase separation process in a binary deformable alloy. It is proved that for a sufficiently regular initial data the trajectory of the solution converges to the ω-limit set of these data. Moreover, it is shown that every element of the ω-limit set is a solution of the corresponding stationary problem.

Keywords: Cahn-Hilliard, viscoelasticity system, phase separation, long-time behaviour.

1. Introduction

We address the issue of the long time behaviour of a system of PDE's describing phase separation process (spinodal decomposition) in binary viscoelastic solids quenched below a critical temperature. The process is driven by thermomechanical effects which lead to microstructure evolution and associated pattern formation. In recent years Cahn-Hilliard systems accounting for elastic effects, known to have a pronounced impact on the phase separation process, have been the subject of many modelling, mathematical and numerical studies, see [1–3] for up-to-date references.

2. Initial-boundary-value problem

The problem under consideration has the form of the following viscoelasticity system coupled with the Cahn-Hilliard equations:

$$\begin{aligned} &\boldsymbol{u}_{tt} - \nabla\cdot[W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) \\ &\quad + \nu\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t)] = \boldsymbol{b} && \text{in } \Omega^\infty = \Omega\times(0,\infty),\\ &\boldsymbol{u}(0)=\boldsymbol{u}_0,\ \ \boldsymbol{u}_t(0)=\boldsymbol{u}_1 && \text{in } \Omega,\\ &\boldsymbol{u}=\boldsymbol{0} && \text{on } S^\infty = S\times(0,\infty), \end{aligned} \tag{1}$$

$$\begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^\infty,\\ &\chi(0)=\chi_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot\nabla\mu = 0 && \text{on } S^\infty, \end{aligned} \tag{2}$$

$$\begin{aligned} &\mu = -\gamma\Delta\chi + \psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) && \text{in } \Omega^\infty,\\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S^\infty, \end{aligned} \tag{3}$$

where $\Omega\subset \mathrm{IR}^3$ is a bounded domain with a smooth boundary S; the unknowns are the fields $\boldsymbol{u}:\Omega^\infty\to\mathrm{IR}^3$, $\chi:\Omega^\infty\to\mathrm{IR}$, and $\mu:\Omega^\infty\to\mathrm{IR}$, representing respectively the displacement vector, the order parameter and the chemical potential; $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ is the linearized strain tensor; functions $W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)$ and $\psi(\chi)$ are specified below, ν, γ are positive constants.

System (1)–(3) represents balance laws of linear momentum, mass, and the equation for the chemical potential. The associated free energy density has the Landau-Ginzburg form

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi,\nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) + \psi(\chi) + \frac{\gamma}{2}|\nabla\chi|^2,$$

where

$$W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) = \frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi))\cdot\boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)),$$

and

$$\psi = \frac{1}{4}(1-\chi^2)^2$$

represent respectively the elastic energy and the double-well potential; positive constant γ is related to the surface tension.
The order parameter χ characterizes the material phase. In case of a binary alloy it is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components. It is assumed that $\chi=-1$ is identified with the phase a and $\chi=1$ with the phase b.
The elasticity tensor $\boldsymbol{A} = (A_{ijkl})$ and the eigenstrain tensor $\bar{\boldsymbol{\varepsilon}}(\chi) = (\bar{\varepsilon}_{ij}(\chi))$ are given by

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bar{\lambda}tr\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{I} + 2\bar{\mu}\boldsymbol{\varepsilon}(\boldsymbol{u}),\\ &\bar{\boldsymbol{\varepsilon}}(\chi) = (1-z(\chi))\bar{\boldsymbol{\varepsilon}}_a + z(\chi)\bar{\boldsymbol{\varepsilon}}_b, \end{aligned}$$

where $\boldsymbol{I}$ is the identity tensor, $\bar{\lambda},\bar{\mu}$ are the Lamé constants satisfying $\bar{\mu}>0$, $3\bar{\lambda}+2\bar{\mu}>0$, $\bar{\boldsymbol{\varepsilon}}_a,\bar{\boldsymbol{\varepsilon}}_b$ are constant eigenstrains of phases a,b, and $z:\mathrm{IR}\to[0,1]$ is a sufficiently smooth interpolation function such that

$$z(\chi)=0 \quad\text{for}\quad \chi\le -1 \quad\text{and}\quad z(\chi)=1 \quad\text{for}\quad \chi\ge 1.$$

The term $\nu\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$, with $\nu = \text{const} > 0$, represents a viscous stress tensor; ν is a viscosity coefficient.

3. Global existence

In [2] we have proved that system (1)–(3) admits a unique global solution $(\boldsymbol{u}, \chi, \mu)$ such that

$$\boldsymbol{u} \in C^1([0,\infty); \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)) \cap C^2([0,\infty); \boldsymbol{H}^1_0(\Omega)),$$
$$\chi \in C([0,\infty); H^2_N(\Omega)) \cap C^1([0,\infty); L_2(\Omega)),$$
$$\mu \in C([0,\infty); H^2_N(\Omega)), \quad \fint_\Omega \chi(t)dx = \chi_m := \fint_\Omega \chi_0 dx$$
$$\text{for all } t \in [0,\infty),$$

for initial data satisfying

$$(\boldsymbol{u}(0), \boldsymbol{u}_t(0), \boldsymbol{u}_{tt}(0), \chi(0), \chi_t(0)) \in \mathcal{W}$$
$$:= \{(\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega))$$
$$\times (\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)) \times \boldsymbol{H}^1_0(\Omega)$$
$$\times H^2_N(\Omega) \times L_2(\Omega)\},$$

where

$$H^2_N(\Omega) = \{\xi : \xi \in H^2(\Omega), \boldsymbol{n} \cdot \nabla\xi = 0 \text{ on } S\},$$
$$\fint_\Omega \chi_0 dx = \frac{1}{|\Omega|} \int_\Omega \chi_0 dx,$$

and $\boldsymbol{u}_{tt}(0) =: \boldsymbol{u}_2$, $\chi_t(0) =: \chi_1$ are calculated in accord with (1)–(3).
Thus, the solution defines the nonlinear, strongly continuous semigroup

$$S(t) : \mathcal{W} \ni (\boldsymbol{u}(0), \boldsymbol{u}_t(0), \boldsymbol{u}_{tt}(0), \chi(0), \chi_t(0)) =: \zeta_0 \mapsto$$
$$\zeta(t) := (\boldsymbol{u}(t), \boldsymbol{u}_t(t), \boldsymbol{u}_{tt}(t), \chi(t), \chi_t(t)) \in \mathcal{W}, \ \ t \in [0,\infty).$$

The proof of the above result is based on the Galerkin method. Firstly, the existence of a solution on a finite time interval is proved. Secondly, the solution is prolonged step by step up to ∞. The crucial role in prolonging play absorbing-type estimates with the property of exponentially time-decreasing impact of the initial data. We use two kinds of such estimates: – the energy estimates derived from the original form of the system, and – the regularity estimates derived from the time-differentiated form of the system.

4. Asymptotic behaviour

In [3] it has been proved that for any initial data belonging to $\mathcal{W}$ the trajectory of the solution converges as $t \to \infty$ to the ω-limit set of these data.

Moreover, it has been shown that the ω-limit set is compact and connected subset of the space

$$\mathcal{Z} := \boldsymbol{H}^1_0(\Omega) \times \boldsymbol{H}^1_0(\Omega) \times \mathbf{L}_2(\Omega) \times H^1(\Omega) \times (H^1(\Omega))',$$

and enjoys the standard properties, namely it is positive invariant with respect to semigroup $S(t)$ defined by the solution and the total energy functional is constant on this set. It also has been proved that every element of the ω-limit set is a solution of the corresponding stationary problem.
Let

$$\omega(\zeta_0) := \{\zeta_\infty = (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t}) \in \mathcal{W} \subset \mathcal{Z} :$$
$$\exists\{t_n\} \subset (0,\infty), t_n \to \infty \text{ and}$$
$$\zeta(t_n) = S(t_n)\zeta_0 \to \zeta_\infty \text{ strongly in } \mathcal{Z}\}.$$

denote the ω-limit set of the initial data $\zeta_0 \in W$.

The main result reads as follows.

Theorem. *[3] Let $\mathcal{S}(t) : \mathcal{W} \to \mathcal{W}$, $t \geq 0$, be the nonlinear semigroup generated by the unique solution of system (1)–(3). Then:*

(i) The ω-limit set $\omega(\zeta_0)$ of the initial data $\zeta_0 = (\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1) \in \mathcal{W} \subset \mathcal{Z}$ is a nonempty, compact and connected subset of the space $\mathcal{Z}$. Furthermore, $\omega(\zeta_0)$ is positive invariant with respect to $S(t)$, i.e.,

$$S(t)\omega(\zeta_0) \subset \omega(\zeta_0) \quad for\ any\ \ t \geq 0;$$

(ii) If $\boldsymbol{b} = \mathbf{0}$ then the map $F_\Omega : \mathcal{W} \to I\!R$, defined by

$$F_\Omega(\zeta(t)) = \int_\Omega \Big[\frac{1}{2}|\boldsymbol{u}_t(t)|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \chi(t))$$
$$+ \psi(\chi(t)) + \frac{\gamma}{2}|\nabla\chi(t)|^2\Big] dx,$$

is the Lyapunov functional for the semigroup $S(t)$, i.e.,

$$F_\Omega(S(t)\zeta_0) \leq F_\Omega(\zeta_0) \quad for\ any\ \ \zeta_0 \in \mathcal{W}, \quad t \geq 0;$$

F_Ω is constant on the ω-limit set $\omega(\zeta_0)$.

(iii) Every element $\zeta_\infty = (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t})$ of the ω-limit set $\omega(\zeta_0)$ is characterized by

$$\zeta_\infty \equiv (\boldsymbol{u}_\infty, \mathbf{0}, \mathbf{0}, \chi_\infty, 0),$$

with functions $\boldsymbol{u}_\infty, \chi_\infty$ independent of time, solving the stationary problem corresponding to (1)–(3):

$$-\nabla \cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty), \chi_\infty) = \mathbf{0} \qquad a.e.\ in\ \ \Omega,$$
$$\boldsymbol{u}_\infty = \mathbf{0} \qquad a.e.\ on\ \ S,$$

$$-\gamma\Delta\chi_\infty + \psi'(\chi_\infty)$$
$$+ W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty), \chi_\infty) = \bar{\mu} \qquad a.e.\ in\ \ \Omega,$$
$$\boldsymbol{n} \cdot \nabla\chi_\infty = 0 \qquad a.e.\ on\ S,$$
$$\fint_\Omega \chi_\infty dx = \chi_m := \fint_\Omega \chi_0 dx,$$

where $\bar{\mu}$ is a constant to be determined along with functions $\boldsymbol{u}_\infty, \chi_\infty$.

In the proof of this theorem the key role play absorbing-type estimates established in the existence proof [2]. Due to such estimates analysis similar to that in [4] is applied.

References

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