Recovering stresses from surface integrals and applications

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Abstract

In the paper we present new formulas for computing strains or stresses at a point in terms of integrals of displacement over the surrounding sphere. They allow us to obtain the regular form of the perturbation of the Steklov-Poincaré operator, which is suitable for analysing the effect of singular domain perturbations on solutions and energy functionals.

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1. Domain decomposition with Steklov-Poincaré operators for linear problems

We study the isotropic elasticity system

$$(1-2\nu)\Delta\mathbf{u} + \mathbf{grad}\,\mathrm{div}\,\mathbf{u} = 0, \text{ in } \Omega \subset \mathbb{R}^3, \qquad (1)$$
$$\mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma_D, \qquad \boldsymbol{\sigma}(\mathbf{u}).\mathbf{n} = \mathbf{h} \text{ on } \Gamma_N,$$

where $\mathbf{u} = (u_1, u_2, u_3)^\top$, ν – Poisson ratio, and $\Gamma_D \cup \Gamma_N = \partial\Omega$. Simultaneously we consider similar problem for $\mathbf{u}_\epsilon$:

$$(1-2\nu)\Delta\mathbf{u}_\epsilon + \mathbf{grad}\,\mathrm{div}\,\mathbf{u}_\epsilon = 0, \text{ in } \Omega(\epsilon), \qquad (2)$$
$$\mathbf{u}_\epsilon = \mathbf{u}_0 \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}(\mathbf{u}_\epsilon).\mathbf{n} = \mathbf{h} \text{ on } \Gamma_N,$$
$$\boldsymbol{\sigma}(\mathbf{u}_\epsilon).\mathbf{n} = 0 \text{ on } \Gamma_\epsilon$$

with $\Omega(\epsilon) = \Omega \setminus \overline{B}(\epsilon)$, $\Gamma_\epsilon = \partial B(\epsilon)$ and $0 \in \mathrm{int}\,\Omega$, so that $B(\epsilon) = \{\, \mathbf{x} \in \mathbb{R}^3 \,|\, \|\mathbf{x}\| < \epsilon \,\} \subset \mathrm{int}\,\Omega$ for ϵ small enough.. Observe that $\Omega(\epsilon)$ is obtained from Ω by singular perturbation changing the topology. Let us define energy functionals

$$I(\boldsymbol{\varphi}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\boldsymbol{\varphi}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,dx - \int_{\Gamma_D} \mathbf{h}.\boldsymbol{\varphi}\,ds,$$

$$I_\epsilon(\boldsymbol{\varphi}_\epsilon) = \frac{1}{2}\int_{\Omega(\epsilon)} \boldsymbol{\sigma}(\boldsymbol{\varphi}_\epsilon) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_\epsilon)\,dx - \int_{\Gamma_D} \mathbf{h}.\boldsymbol{\varphi}_\epsilon\,ds.$$

Then the solutions may be characterized by

$$I(\mathbf{u}) = \inf_{\boldsymbol{\varphi}\in V} I(\boldsymbol{\varphi}), \quad I_\epsilon(\mathbf{u}_\epsilon) = \inf_{\boldsymbol{\varphi}_\epsilon \in V_\epsilon} I_\epsilon(\boldsymbol{\varphi}_\epsilon), \qquad (3)$$

where

$$V = \{\, \boldsymbol{\varphi} \in \mathbf{H}^1(\Omega) \,|\, \boldsymbol{\varphi} = \mathbf{u}_0 \text{ on } \Gamma_D \,\},$$

$$V_\epsilon = \{\, \boldsymbol{\varphi}_\epsilon \in \mathbf{H}^1(\Omega(\epsilon)) \,|\, \boldsymbol{\varphi}_\epsilon = \mathbf{u}_0 \text{ on } \Gamma_D \,\}.$$

For contact problems the minimization in (3) would be carried out over some cone $K \subset V$, $K \subset V_\epsilon$ and $\partial\Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_C$, where Γ_C - potential contact zone. These minimizations correspond then to variational inequalities.

We know [2, 3] that

$$I_\epsilon(\mathbf{u}_\epsilon) = I(\mathbf{u}) - \frac{4}{3}\pi\epsilon^3 e(0) + o(\epsilon^3) \qquad (4)$$

where

$$e(0) = \frac{1}{2}\boldsymbol{\sigma}(\mathbf{u}(0)) : \boldsymbol{\varepsilon}(\mathbf{u}(0)) \qquad (5)$$

is the bulk energy density at $\mathbf{x} = 0$. However, the operator $\mathbf{u} \longmapsto e(0)$ considered as taking point values of stresses is not sufficiently regular for our purposes.

Therefore we surround 0 with $B(R) \subset \mathrm{int}\,\Omega$ and assume ϵ small enough, so that $B(\epsilon) \subset B(R)$. Denote $C(R,\epsilon) = B(R) \setminus \overline{B}(\epsilon)$, $\Gamma_R = \partial B(R)$. Then it is possible to define the Steklov-Poincaré operator $\mathbf{A}_\epsilon : \mathbf{H}^{1/2}(\Gamma_R) \longmapsto \mathbf{H}^{-1/2}(\Gamma_R)$ by means of the BVP:

$$(1-2\nu)\Delta\mathbf{w} + \mathbf{grad}\,\mathrm{div}\,\mathbf{w} = 0, \text{ in } C(R,\epsilon), \qquad (6)$$
$$\mathbf{w} = \mathbf{v} \text{ on } \Gamma_R, \quad \boldsymbol{\sigma}(\mathbf{w}).\mathbf{n} = 0 \text{ on } \Gamma_\epsilon$$

so that

$$A_\epsilon\mathbf{v} = \boldsymbol{\sigma}(\mathbf{w}).\mathbf{n} \text{ on } \Gamma_R. \qquad (7)$$

Let $\mathbf{u}^R$ be the restriction of $\mathbf{u}$ to $\Omega(R)$ and $\gamma^R\boldsymbol{\varphi}$ the projection of $\boldsymbol{\varphi}$ on Γ_R. We may then define the functional

$$I_\epsilon^R(\boldsymbol{\varphi}_\epsilon) = \frac{1}{2}\int_{\Omega(R)} \boldsymbol{\sigma}(\boldsymbol{\varphi}_\epsilon) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_\epsilon)\,dx - \int_{\Gamma_D} \mathbf{h}.\boldsymbol{\varphi}_\epsilon\,ds + \qquad (8)$$
$$+ \frac{1}{2}\int_{\Gamma_R} (\mathbf{A}_\epsilon\gamma^R\boldsymbol{\varphi}_\epsilon).\gamma^R\boldsymbol{\varphi}_\epsilon\,ds$$

and the solution $\mathbf{u}_\epsilon^R$ as a minimal argument for

$$I_\epsilon^R(\mathbf{u}_\epsilon^R) = \inf_{\boldsymbol{\varphi}_\epsilon \in K \subset V_\epsilon} I_\epsilon^R(\boldsymbol{\varphi}_\epsilon), \qquad (9)$$

Here lies the essence of domain decomposition: we have replaced the the variable domain by a fixed one, at the price of introducing variable boundary operator. The goal is to find the expansion

$$\mathbf{A}_\epsilon = \mathbf{A} + \epsilon^3\mathbf{B} + \mathbf{R}_\epsilon \qquad (10)$$

where the remainder $\mathbf{R}_\epsilon$ is of order $o(\epsilon^3)$ in the operator norm $L(\mathbf{H}^{1/2}(\Gamma_R), \mathbf{H}^{-1/2}(\Gamma_R))$, and the operator $\mathbf{B}$ is regular enough, namely it is bounded and linear:

$$\mathbf{B} \in L\big(\mathbf{L}_2(\Gamma_R), \mathbf{L}_2(\Gamma_R)\big).$$

Under this assumption the following propositions hold.

Proposition 1.1 *Assume that* (10) *holds in the operator norm. Then the strong convergence takes place*

$$\mathbf{u}_\epsilon^R \to \mathbf{u}^R \tag{11}$$

in the norm of $\mathbf{H}^1(\Omega(R))$.

Proposition 1.2 *The energy functional has the representation*

$$I_\varepsilon^R(\mathbf{u}_\epsilon^R) = I^R(\mathbf{u}^R) + \epsilon^3 \langle \mathbf{B}(\mathbf{u}^R), \mathbf{u}^R \rangle_R + o(\varepsilon^3)\ , \tag{12}$$

where $o(\varepsilon^3)/\varepsilon^3 \to 0$ *with* $\varepsilon^3 \to 0$ *in the same energy norm.*

Here $I^R(\mathbf{u}^R)$ denotes the functional I_ϵ^R on intact domain, i.e. $\epsilon := 0$ and $\mathbf{A}_\epsilon := \mathbf{A}$, applied to truncation of $\mathbf{u}$.

This approach is important for variational inequalities since it allows us to derive the formulas for topological derivatives which coincide with the formulas obtained for the corresponding linear boundary value problems.

2. Explicit form of the operator B

Assuming given values of $\mathbf{u}$ on Γ_R, the solution of elasticity system in $B(R)$ may be expressed, following partially the derivation from [1], as

$$\mathbf{u} = \sum_{n=0}^{\infty} [\mathbf{U}_n + (R^2 - r^2) k_n(\nu) \mathbf{grad}\, \mathrm{div}\, \mathbf{U}_n]. \tag{13}$$

where $k_n(\nu) = 1/2[(3-2\nu)n - 2(1-\nu)]$ and $r = \|\mathbf{x}\|$. In addition

$$\mathbf{U}_n = \frac{1}{R^n}[\mathbf{a}_{n0} d_n(\mathbf{x}) + \sum_{m=1}^{n} (\mathbf{a}_{nm} c_n^m(\mathbf{x}) + \mathbf{b}_{nm} s_n^m(\mathbf{x}))]. \tag{14}$$

The vectors

$$\mathbf{a}_{n0} = (a_{n0}^1, a_{n0}^2, a_{n0}^3)^\top,$$
$$\mathbf{a}_{nm} = (a_{nm}^1, a_{nm}^2, a_{nm}^3)^\top,$$
$$\mathbf{b}_{nm} = (b_{nm}^1, b_{nm}^2, b_{nm}^3)^\top$$

are constant and the set of functions

$$\{d_0;\ d_1, c_1^1, s_1^1;\ d_2, c_2^1, s_2^1, c_2^2, s_2^2;\ d_3, c_3^1, s_3^1, c_3^2, s_3^2, c_3^3, s_3^3;\ \ldots\}$$

constitutes the complete system of orthonormal harmonic polynomials on Γ_R, related to Laplace spherical functions. They are explicitly known, e.g.

$$c_3^2(\mathbf{x}) = \frac{1}{R^4}\sqrt{\frac{7}{240\pi}}(15x_1^2x_3 - 15x_2^2x_3),$$

and n indicates always the order of the polynomial. If the value of $\mathbf{u}$ on Γ_R is assumed as given, then, denoting

$$\langle \phi, \psi \rangle_R = \int_{\Gamma_R} \phi\psi\, ds,$$

we have for $n \geq 0$, $m = 1..n$, $i = 1, 2, 3$:

$$a_{n0}^i = R^n \langle u_i, d_n(\mathbf{x}) \rangle_R, \tag{15}$$
$$a_{nm}^i = R^n \langle u_i, c_n^m(\mathbf{x}) \rangle_R,$$
$$b_{nm}^i = R^n \langle u_i, s_n^m(\mathbf{x}) \rangle_R.$$

Since we are looking for $\varepsilon_{ij}(0)$, only the part of $\mathbf{u}$ which is linear in $\mathbf{x}$ is relevant. It contains two terms:

$$\hat{\mathbf{u}} = \mathbf{U}_1 + R^2 k_3(\nu) \mathbf{grad}\, \mathrm{div}\, \mathbf{U}_3. \tag{16}$$

For any $f(\mathbf{x})$, $\mathbf{grad}\, \mathrm{div}\, (\mathbf{a}f) = H(f) \cdot \mathbf{a}$, where $H(f)$ is the Hessian matrix of f. Therefore

$$\begin{aligned}\hat{\mathbf{u}} = {} & \frac{1}{R}[\mathbf{a}_{10} d_1(\mathbf{x}) + \mathbf{a}_{11} c_1^1(\mathbf{x}) + \mathbf{b}_{11} s_1^1(\mathbf{x}))] \\ & + R^2 k_3(\nu) \frac{1}{R^3}\Big[H(d_3)(\mathbf{x})\mathbf{a}_{30} \\ & + \sum_{m=1}^{3}\big(H(c_3^m)(\mathbf{x})\mathbf{a}_{3m} + H(s_3^m)(\mathbf{x})\mathbf{b}_{3m}\big)\Big]\end{aligned} \tag{17}$$

From the above we may single out the coefficients standing at x_1, x_2, x_3 in u_1, u_2, u_3. For example,

$$\begin{aligned}\varepsilon_{11}(0) = {} & \frac{1}{R^3}\sqrt{\frac{3}{4\pi}} a_{11}^1 + \frac{1}{R^5} k_3(\nu)\Big[-3\sqrt{\frac{7}{4\pi}} a_{30}^3 \\ & - 9\sqrt{\frac{7}{24\pi}} a_{31}^1 - 3\sqrt{\frac{7}{24\pi}} b_{31}^2 + 30\sqrt{\frac{7}{240\pi}} a_{32}^3 \\ & + 90\sqrt{\frac{7}{1440\pi}} a_{33}^1 + 90\sqrt{\frac{7}{1440\pi}} b_{33}^2\Big], \\ \varepsilon_{12}(0) = {} & \frac{1}{R^3}\sqrt{\frac{3}{4\pi}}(b_{11}^1 + a_{11}^2) + \frac{1}{R^5} k_3(\nu)\Big[-3\sqrt{\frac{7}{24\pi}} a_{31}^2 \\ & - \sqrt{\frac{7}{24\pi}} b_{31}^1 + 15\sqrt{\frac{7}{60\pi}} b_{32}^3 \\ & - 90\sqrt{\frac{7}{1440\pi}} a_{33}^2 + 90\sqrt{\frac{7}{1440\pi}} b_{33}^1\Big].\end{aligned} \tag{18}$$

These formulas are exact. As a result, the operator $\mathbf{B}$ may be defined by the formula

$$\langle \mathbf{Bu}, \mathbf{u} \rangle_R = -\frac{2}{3}\pi \boldsymbol{\sigma}(\mathbf{u}(0)) : \boldsymbol{\varepsilon}(\mathbf{u}(0))$$

but the right-hand side consists of integrals of $\mathbf{u}$ over Γ_R resulting from (15). Thus the new expressions (18) for strains make possible to rewrite $\mathbf{B}$ in the form possessing the desired regularity.

References

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