Vector minimization of the compliance of elastic anisotropic plates

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Abstract

The paper deals with optimal design of linearly elastic plates of the Kelvin moduli being distributed according to a given pattern. The case of two loading conditions is discussed. The optimal plate is characterized by the minimum value of the weighted sum of the compliances corresponding to the two kinds of loads. The problem is reduced to the equilibrium problem of a hyper-elastic mixture of properties expressed in terms of two stress fields. This highly non-linear problem, expressed in terms of two displacement fields, is solved numerically by the finite element method, using special Newton's solver. Exemplary numerical results are presented delivering layouts of variation of elastic characteristics for selected values of the weighting factors corresponding to two kinds of loadings.

Keywords: anisotropy, elasticity, finite element methods, material properties, numerical analysis, Pareto optimization, plates

1. Introduction

The problem of maximization of the overall stiffness of a structure can be formulated in several manners. One can assume that two or three materials are at our disposal, the amount of both being given. The aim is to find the optimal placement of these materials. Majority of papers in this vein are targeted at the shape design and base on the relaxation by homogenization method. The alternative method is FMD (free material design approach) where all the anisotropic properties are treated as design variables of non-uniform distribution, see e.g. chapter 3 in the book [1]. The Hooke tensor encompasses all these properties. In FMD no point-wise conditions are imposed on this tensor, instead an integral isoperimetric condition is assumed. This approach is still developed. The essential drawback of the FMD approach is the unclear physical meaning of the isoperimetric condition. The other drawback of FMD is the very result which leads to one non-zero elastic modulus. This degeneracy is a direct consequence of the formulation in which the moduli must follow the one stress field. The present paper puts forward a new version of the FMD in which the Kelvin moduli are viewed as distributed according to a given pattern. The other elastic characteristics – called eigen-states – are design variables. We refer here to the theory of spectral decomposition of Hooke's tensor developed in [2]. No integral type isoperimetric condition is imposed. Two loading conditions are considered. The merit function is the weighted sum of the total compliances corresponding to two loading conditions. Introduction of two loading states will make the distribution of Hooke tensor unique, of lesser degeneracy than in the one loading case. Just this is the main reason to consider two types of loads. The departure point is stress-based. This makes the formulation a minimum problem with respect to: statically admissible stress fields and design variables. The problem discussed brings about local minimization problems which turn out to be solvable by relatively elementary methods. By solving the local problems explicitly, the main problem is rearranged to the form of an equilibrium problem of a hypothetic hyper-elastic body composed of two constituents, within a mixture theory. The constitutive equations couple two strain fields with two stress fields in a nonlinear manner. These equations are inverted to the primal form, which makes it possible to recover the strain-based potential. This effective potential turns out to be convex thus corresponding to monotonicity of the stress-strain equations and to solvability of the optimization problem considered. The FEM is applied using quadrilateral elements with bilinear shape functions. We show that a special Newton type algorithm is a right tool to solve a rich family of optimization problems for various loading cases and contrasts between the Kelvin moduli.

2. Formulation of an optimal design

Consider a thin elastic plate loaded in plane. Assume that the plane stress assumptions are fulfilled. The plate is subject to a boundary loading $\overset{\alpha}{\mathbf{T}}\left(\alpha=1,2\right)$ acting on a part Γ_1 of the boundary $\partial\Omega$ of a given plane domain Ω, parameterized by Cartesian coordinates $\left(x_1,x_2\right)$ with the orthonormal basis $\left\{\mathbf{e}_i\right\}_{i=1,2}$. The plate is fixed along Γ_2 being a part of $\partial\Omega$, i.e. $\overset{\alpha}{\mathbf{u}}\big|_{\Gamma_2}=\mathbf{0}$, where $\overset{\alpha}{\mathbf{u}}=(\overset{\alpha}{u}_1,\overset{\alpha}{u}_2)$ are the unknown displacement fields referred to Ω – the response of the plate to the boundary loading $\overset{\alpha}{\mathbf{T}}$. Let $\mathbf{C}$ be the tensor of elastic moduli of the plane stress problem, i.e. $\overset{\alpha}{\boldsymbol{\sigma}}=\mathbf{C}\overset{\alpha}{\boldsymbol{\varepsilon}}$, where $\overset{\alpha}{\boldsymbol{\sigma}}=(\overset{\alpha}{\sigma}_{ij})$, $\overset{\alpha}{\boldsymbol{\varepsilon}}=(\overset{\alpha}{\varepsilon}_{ij})$ (*i,j*=1,2) are respectively the symmetric stress tensors and symmetric strain tensors $\overset{\alpha}{\boldsymbol{\varepsilon}}=(\nabla\overset{\alpha}{\mathbf{u}}+\nabla\overset{\alpha}{\mathbf{u}}^T)/2$ being the symmetric part of the displacement gradients $\nabla\overset{\alpha}{\mathbf{u}}=(\partial\overset{\alpha}{u}_i/\partial x_j)$. At each point $x\in\Omega$ tensor $\mathbf{C}=\sum_{K=1}^{3}\lambda_K\mathbf{P}_K$, $\mathbf{C}^{-1}=\sum_{K=1}^{3}\lambda_K^{-1}\mathbf{P}_K$ is characterized by its eigenvalues $\lambda_K\left(x\right)$ called Kelvin moduli and the corresponding projection operators $\mathbf{P}_K=\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K$ being tensorial products of so called eigenstates $\boldsymbol{\omega}_K\left(x\right)\left(K=1,2,3\right)$. We shall assume, see e.g. [2], that $\lambda_1\geq\lambda_2\geq\lambda_3>0$ and $\boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L=\delta_{KL}$. Let us consider the equilibrium problem of the plate subject to the loading of index α :

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$$\overset{\alpha}{\mathbb{C}} = \min_{\text{statically admissible } \boldsymbol{\tau}} \left(\frac{1}{2} \int_{\Omega} \boldsymbol{\tau} \cdot \mathbf{C}^{-1} \boldsymbol{\tau} \, dx \right) \text{ where } \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0},\ \boldsymbol{\tau}\big|_{\Gamma_1} \mathbf{n} = \overset{\alpha}{\mathbf{T}}$$

with $\mathbf{n} \in \mathbb{R}^2$ as a normal to $\partial\Omega$. It is well known, that the optimal value $\overset{\alpha}{\mathbb{C}}$ for the solution $\overset{\alpha}{\boldsymbol{\sigma}}$ is equal to the value of the compliance of the plate in the equilibrium state. Let us notice that the optimal value $\overset{\alpha}{\mathbb{C}}$ is dependent, via $\mathbf{C}^{-1}(x)$, of the Kelvin moduli $\lambda_K(x)$ and eigenstates $\boldsymbol{\omega}_K(x)$. Assuming that the distribution of the Kelvin moduli $\lambda_K(x)$ within Ω is viewed as prescribed (they do not undergo optimization), independently of the minimization of the compliance functional $\min_{\text{statically admissible } \boldsymbol{\tau}} \left(\frac{1}{2} \int_{\Omega} \boldsymbol{\tau} \cdot \mathbf{C}^{-1} \boldsymbol{\tau} \, dx \right)$ it is possible additionally to minimize the $\overset{\alpha}{\mathbb{C}} = \overset{\alpha}{\mathbb{C}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ with respect to the eigenstates $\boldsymbol{\omega}_K$ fulfilling the ortonormality conditions $\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}$, $\|\boldsymbol{\omega}_K\| = 1$. Let $\eta \in [0,1]$ be a fixed number. Let us introduce the cost function of our problem $F_\eta(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) = \eta \overset{1}{\mathbb{C}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) + (1-\eta) \overset{2}{\mathbb{C}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ being a linear combination of the compliances corresponding to two kinds of loadings: $\overset{1}{\mathbf{T}}$ and $\overset{2}{\mathbf{T}}$ acting on Γ_1 and to the same distribution of $\boldsymbol{\omega}_K$. It is not difficult to show that $\overset{\alpha}{\mathbb{C}} = \overset{\alpha}{\mathbb{C}}(\boldsymbol{\omega}_2, \boldsymbol{\omega}_3) = \min_{\text{statically admissible } \boldsymbol{\tau}} \left[\int_{\Omega} k(\boldsymbol{\tau}, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) dx \right]$ where

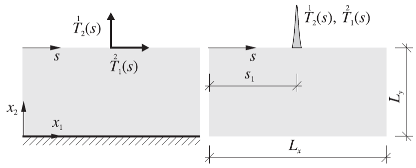
$$k(\boldsymbol{\tau}, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) = \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 + \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) (\boldsymbol{\omega}_2 \cdot \boldsymbol{\tau})^2 + \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_1} \right) (\boldsymbol{\omega}_3 \cdot \boldsymbol{\tau})^2,$$

so we can assume that $F_\eta = F_\eta(\boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$. Assume that $Q = \{(\mathbf{a}, \mathbf{b}) \in Sym \times Sym;\ \mathbf{a} \cdot \mathbf{b} = 0 \wedge \mathbf{a} \cdot \mathbf{a} = 1 \wedge \mathbf{b} \cdot \mathbf{b} = 1\}$ and let $Q(\Omega)$ be the set of tensor fields **a**, **b** from *Sym* given on Ω which for almost all $x \in \Omega$ satisfy $(\mathbf{a}(x), \mathbf{b}(x)) \in Q$ (*Sym* denotes the set of symmetric tensors). Our goal is to solve the family of minimization problems $I_\eta = \min_{(\boldsymbol{\omega}_2, \boldsymbol{\omega}_3) \in Q(\Omega)} F_\eta(\boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$. It is possible to show (the proof is not trivial) that the above problem is equivalent to the following one: for given $\eta \in [0,1]$ find the fields $\overset{1}{\mathbf{u}}, \overset{1}{\boldsymbol{\varepsilon}}, \overset{1}{\boldsymbol{\sigma}}$ and $\overset{2}{\mathbf{u}}, \overset{2}{\boldsymbol{\varepsilon}}, \overset{2}{\boldsymbol{\sigma}}$ in Ω such that $\overset{\alpha}{\mathbf{u}}\big|_{\Gamma_2} = \mathbf{0}$, $\overset{\alpha}{\boldsymbol{\varepsilon}} = \frac{1}{2}(\nabla \overset{\alpha}{\mathbf{u}} + \nabla \overset{\alpha}{\mathbf{u}}^T)$, $\overset{1}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overset{1}{\boldsymbol{\varepsilon}}, \overset{2}{\boldsymbol{\varepsilon}})$, $\overset{2}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overset{2}{\boldsymbol{\varepsilon}}, \overset{1}{\boldsymbol{\varepsilon}})$ $(\alpha = 1,2)$ and $\int_{\Omega} \overset{1}{\boldsymbol{\sigma}} \cdot \mathbf{e}\, dx = \int_{\Gamma_1} \overset{1}{\mathbf{T}} \cdot \mathbf{v}\, ds$, $\int_{\Omega} \overset{2}{\boldsymbol{\sigma}} \cdot \mathbf{e}\, dx = \int_{\Gamma_1} \overset{2}{\mathbf{T}} \cdot \mathbf{v}\, ds$ for all kinematically admissible **v** i.e. $\mathbf{v}\big|_{\Gamma_2} = \mathbf{0}$ and $\mathbf{e} = (\nabla \mathbf{v} + \nabla \mathbf{v}^T)/2$. The stress fields $\overset{1}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overset{1}{\boldsymbol{\varepsilon}}, \overset{2}{\boldsymbol{\varepsilon}})$, $\overset{2}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overset{2}{\boldsymbol{\varepsilon}}, \overset{1}{\boldsymbol{\varepsilon}})$ are linked with $\overset{1}{\boldsymbol{\varepsilon}}, \overset{2}{\boldsymbol{\varepsilon}}$ by the highly nonlinear and too long to be shown here formulae. Having found the fields $\overset{1}{\boldsymbol{\sigma}}, \overset{2}{\boldsymbol{\sigma}}$ we can immediately calculate in Ω the values of the eigenstates $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ and find optimal layout of the components $C_{ijkl} = \sum_{K=1}^{3} \lambda_K \omega_{Kij} \omega_{Kkl}$ of the optimal elasticity tensor $\mathbf{C} = C_{ijkl}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$.

3. Numerical treatment of the problem

The formulated problem has not its counterpart in the literature. The degree of complexity of this problem resembles that of the equilibrium of strongly curved thin hyper-elastic shells, where two stress resultant fields and two strain fields are involved. Moreover, the constitutive equations are nonlinear and coupled; their form is specific and they have never been implemented in the numerical analysis. The above features of the problem discussed justify providing the details of the finite element method applied (for obvious reasons not given in this short presentation).

An exemplary case study concerns the plate clamped along the lower edge (x_2=0), subject either to a vertical load $\overset{1}{\mathbf{T}} = (0, \overset{1}{T}_2)$ at its top edge or to the horizontal load $\overset{2}{\mathbf{T}} = (\overset{2}{T}_1, 0)$ at the same edge. The tractions $\overset{1}{T}_2, \overset{2}{T}_1$ are modeled by specifically chosen weight functions.



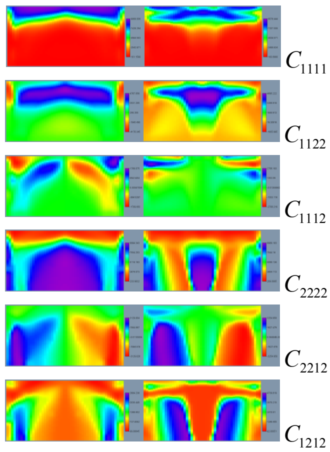


Figure 1: Distribution of the optimal moduli C_{ijkl} for η = 0.9 (first column) and η = 0.1 (second column)

The minimization procedure leads to the optimal distribution of all components of the Hooke tensor for selected values of the weighting factor η, cf Fig.1. The optimal plate turns out to be highly anisotropic, exhibiting auxetic properties in the regions, where the counterparts of the Poisson ratios assume negative values.

References

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