Modeling and analysis of rate-independent damage and delamination processes

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Abstract

This contribution deals with the modeling of damage and delamination phenomena as rate-independent processes. Different models are presented and their analysis using the energetic formulation of rate-independent processes is commented.

Keywords: partial isotropic damage, gradient delamination, Griffith-type delamination, rate-independence, energetic formulation.

1. Introduction

Damage means the creation and growth of cracks and voids on the micro-level of a solid material. Although it still may be observed as a continuous body on the macroscopic level the evolution of micro-defects may change the mechanical properties of the solid. This phenomenon can be described by means of continuum damage mechanics, which was introduced by L.M. Kachanov in 1958, see e.g. [1]. Within this approach an inner variable, the damage variable $z:[0,T]\times\Omega\to[0,1]$, is incorporated to the constitutive law, where it reflects the changes in the elastic behavior of the body due to damage. Here, $[0,T]$ denotes a time interval and $\Omega\subset\mathbb{R}^d$, $d\in\mathbb{N}$, the reference domain of the body. Moreover, $z(t,x)=1$ stands for no damage and $z(t,x)=0$ for complete damage in the point $(t,x)\in[0,T]\times\Omega$. With similar ideas also the delamination, i.e. the micro-cracking, of a compound along an interface Γ_C can be described. Then, the inner variable $z:[0,T]\times\Gamma_\mathrm{C}\to[0,1]$ denotes the delamination variable, which accounts for the constitution of the bonding along the interface.

In many materials the healing of defects is impossible. With the above notion the unidirectionality of the defect evolution is described by the condition

$$\dot z\le 0\,, \tag{1}$$

where $\dot z$ is the partial time-derivative of the inner variable z.

For various applications the temporal evolution of defects can be considered as rate-independent. This is the case if the time scales imposed to the system from the exterior are much larger than the intrinsic ones, i.e. if the external loadings evolve much slower than the inner variables.

This contribution deals with the modeling of damage and delamination in the rate-independent setting. Section 2 describes the mathematical modeling in the framework of the energetic formulation of rate-independent processes. In Section 3 a model for partial isotropic damage is presented. Section 3.1 discusses the analytical tools and challenges in the analysis of this model. Section 3.2 gives an overview over existence results for different aspects in the mathematical modeling. Examples from engineering which apply to this theory are given in the Sections 3.3–3.7. Section 4 presents several delamination models. They can be derived as the limits of damage models. The analytical tool for this limit passage is the Γ-convergence of rate-independent processes, which is briefly introduced in Section 4.1. The paper concludes with a summary of the presented results in Section 5.

2. Energetic formulation of rate-independent processes

In the rate-independent setting the evolution of defects can be described mathematically using the so-called energetic formulation of the process. This approach is solely based on an energy functional $\mathcal{E}:[0,T]\times\mathcal{Q}\to\mathbb{R}\cup\{\infty\}$ and on a dissipation potential $\mathcal{R}:\mathcal{Z}\to[0,\infty]$ which takes into account the evolution of the inner variable. Thereby $\mathcal{Q}=\mathcal{U}\times\mathcal{Z}$ is the state space with $\mathcal{U}$ as the set of admissible displacements and $\mathcal{Z}$ as the set of admissible inner variables. Rate-independence is featured by the positive 1-homogeneity of $\mathcal{R}$, i.e. $\mathcal{R}(0)=0$ and $\mathcal{R}(\alpha v)=\alpha\mathcal{R}(v)$ for all $\alpha\in(0,\infty]$ and all $v\in\mathcal{Z}$. Moreover, (1) is incorporated to the model by using the following form for $\mathcal{R}$:

$$\mathcal{R}(\dot z):=\int_D R(\dot z(x))\,\mathrm{d}x \text{ with } R(\dot z):=\begin{cases}\varrho|\dot z| & \text{if } \dot z\le 0\,,\\ \infty & \text{otherwise}\,,\end{cases} \tag{2}$$

for a constant $\varrho>0$ and $D\in\{\Omega,\Gamma_\mathrm{C}\}$.

The notion of solution used in the energetic formulation is the so-called energetic solution $q=(u,z):[0,T]\to\mathcal{Q}$, which is characterized by satisfying the global stability condition (S) and the global energy balance (E) for all $s,t\in[0,T]$:

$$\begin{aligned}&\text{for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q} \text{ holds :}\\ &\qquad\mathcal{E}(t,q(t))\le\mathcal{E}(t,\tilde q)+\mathcal{R}(\tilde z-z(t))\,,\end{aligned} \tag{S}$$

$$\mathcal{E}(t,q(t))+\mathrm{Diss}_\mathcal{R}(z,[s,t])=\mathcal{E}(s,q(s))+\int_s^t\partial_\xi\mathcal{E}(\xi,q(\xi))\,\mathrm{d}\xi \tag{E}$$

with $\mathrm{Diss}_\mathcal{R}(z,[s,t]):=\sup_P\sum_{j=1}^N\mathcal{R}(z(\xi_j)-z(\xi_{j-1}))$ and $P=\{s=\xi_0<\ldots<\xi_N=t,\,N\in\mathbb{N}\}$. In particular, a given initial datum $q_0=(u_0,z_0)\in\mathcal{Q}$ has to satisfy (S) at time $t=0$.

If $\mathcal{E}(t,\cdot)$ is convex and sufficiently smooth and if an energetic solution q is sufficiently smooth in time then the energetic formulation is equivalent to the subdifferential formulation:

$$\text{For all } t\in[0,T]:\ 0\in\partial\mathcal{R}(\dot q(t))+\mathrm{D}_q\mathcal{E}(t,q(t)),\ \ q(0)=q_0\,.$$

Here, $\mathrm{D}_q\mathcal{E}$ denotes the Gâteaux derivative of $\mathcal{E}$ and $\partial\mathcal{R}$ is the (multivalued) subdifferential of the convex, 1-homogeneous, but

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non-differentiable functional $\mathcal{R}$, e.g. as in (2). In general, the energetic formulation provides a very low temporal regularity of energetic solutions, only. In particular, energetic solutions may even jump with respect to time. But under stronger convexity assumptions on the energy functional it can be shown that the temporal regularity of solutions improves to continuity or even to Hölder or Lipschitz continuity, see Section 3.6 and [2, 3].

3. A model for partial isotropic damage

In the following a model describing the partial isotropic damage in nonlinearly elastic materials is presented. Thereby, partial damage means that the material cannot completely disintegrate, i.e. there is a lower bound $z_\star > 0$ for the damage variable and for $z = z_\star$ the material is still able to support arbitrary stresses without further damage. Based on a damage model proposed in [4] the energy functional is assumed to be of the following form

$$\mathcal{E}(t,u,z) := \int_\Omega W(e(u),z)\,\mathrm{d}x - \langle l(t),u\rangle + \mathcal{G}(z)\,. \tag{4}$$

The rate-independent damage process is driven by the time-dependent external loadings $l : [0,T] \to \mathcal{U}^*$ (second term), where $\mathcal{U}^*$ stands for the dual space of the Banach space $\mathcal{U}$. The first term in (4) denotes the stored elastic energy of the body. In many engineering contributions there is made an ansatz for W, which links the damage variable and the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ multiplicatively [5], such as e.g.

$$W(e,z) = z\mu|e|^2 + \tfrac{\lambda}{2}|\operatorname{tr} e|^2\,. \tag{5}$$

Here $\lambda, \mu > 0$ are the Lamé constants and $\operatorname{tr} e$ is the trace of e. The term $\mathcal{G}(z)$ in (4) involves the gradient of the damage variable. Hence, on the one hand it mathematically serves as a regularization. On the other hand, engineers use the damage gradient to account for nonlocal microscopic interactions [6, 7].

3.1. Analytical tools and challenges

Like damage, also the evolution of other mechanical processes in solids, such as elasto-plastic deformations, phase transitions in shape memory alloys or crack propagation, can be modeled rate-independently, see e.g. [8, 9, 10, 11, 12]. All these processes can be described in terms of an energy functional $\mathcal{E}$ and a positively 1-homogeneous dissipation potential $\mathcal{R}$, so that the energetic formulation (S) & (E) applies. Within the works [13, 14, 15, 16] an abstract existence theory for energetic solutions of rate-independent processes has been developed. First of all, it uses the fact that a convex, positively 1-homogeneous dissipation potential $\mathcal{R}$ satisfies the triangle inequality, namely for all $z_1, z_2, z_3 \in \mathcal{Z}$ we find

$$\begin{aligned}\mathcal{R}(z_1 - z_2) &= 2\mathcal{R}\big(\tfrac{1}{2}(z_1 - z_3) + \tfrac{1}{2}(z_3 - z_2)\big)\\ &\le 2\big(\tfrac{1}{2}\mathcal{R}(z_1 - z_3) + \tfrac{1}{2}\mathcal{R}(z_3 - z_2)\big) = \mathcal{R}(z_1 - z_3) + \mathcal{R}(z_3 - z_2)\,,\end{aligned}$$

where we exploited the 1-homogeneity in the first line and convexity in the second line. Hence the dissipation potential generates a *dissipation distance* on $\mathcal{Z} \times \mathcal{Z} \to [0,\infty]$. If additionally, for all $z_1, z_2 \in \mathcal{Z}$, $\mathcal{R}(z_1 - z_2) = 0$ implies that $z_1 = z_2$, then one speaks of a *quasi-distance*. This notion expresses that two of the axioms of a distance are satisfied, namely the triangle inequality and positivity, but symmetry must not hold and the value ∞ can be attained. This is the case for $\mathcal{R}$ from (2), because for $z_2 \le z_1$ it is $\mathcal{R}(z_1 - z_2) < \infty$, whereas $\mathcal{R}(z_2 - z_1) = \infty$.

The abstract existence theorem [17, Th. 3.4] is based on the assumptions that

$$\mathcal{R} : \mathcal{Z} \to [0,\infty] \text{ generates a quasi-distance,} \tag{6a}$$

$$\mathcal{R} : \mathcal{Z} \to [0,\infty] \text{ is (weakly seq.) lower semicontinuous.} \tag{6b}$$

Moreover, it uses the following assumptions on the energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$

Compactness of energy sublevels: $\forall\, t\in[0,T]\ \forall\, E\in\mathbb{R}$: (7a)
$L_E(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \le E\}$ is weakly seq. compact.

Uniform control of the power: (7b)
$\exists\, c_0\in\mathbb{R}\ \exists\, c_1>0\ \forall\, (t_q,q)\in[0,T]\times\mathcal{Q}$ with $\mathcal{E}(t_q,q) < \infty$:
$\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T])$ and
$|\partial_t\mathcal{E}(t,q)| \le c_1(c_0 + \mathcal{E}(t,q))$ for all $t\in[0,T]$.

These properties ensure the following existence result for energetic solutions of rate-independent processes.

Theorem 1 (Abstract main existence theorem [17, Th. 3.4]) *Let $\mathcal{Q}$ be a closed subset of a reflexive Banach space and let $(\mathcal{Q},\mathcal{E},\mathcal{R})$ satisfy conditions* (6) *and* (7). *Moreover, let the following compatibility conditions hold: For every sequence $(t_k,q_k)_{k\in\mathbb{N}}$ with $(t_k,q_k) \rightharpoonup (t,q)$ in $[0,T]\times\mathcal{Q}$ and (t_k,q_k) satisfying* (S) *for all $k\in\mathbb{N}$ we have*

$$\partial_t\mathcal{E}(t,q_k) \to \partial_t\mathcal{E}(t,q)\,, \tag{8a}$$

$$(t,q) \text{ satisfies (S)}\,. \tag{8b}$$

Then, for each initial datum $(t{=}0,q_0)$ satisfying (S) *there exists an energetic solution $q : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q},\mathcal{E},\mathcal{R})$ with $q(0){=}q_0$.*

The proof of Theorem 1 is based on a time-discretization, where conditions (7a), (6b) ensure the existence of a minimizer for the time-incremental minimization problem at each time-step. For this, the direct method of the calculus of variations as applied. In particular conditions (7a) and (6b) can be verified if $\mathcal{E}$ and $\mathcal{R}$ are convex and coercive. Hence, for a given partition $\Pi := \{0 = t_0 < t_1 < \ldots < t_M = T\}$, for every $k = 1,\ldots,M$ one has to find

$$q_k \in \operatorname{Argmin}\{\mathcal{E}(t_k,\tilde q) + \mathcal{R}(\tilde z - z_{k-1}) \,|\, \tilde q = (\tilde u,\tilde z) \in \mathcal{Q}\}\,. \tag{9}$$

One then defines a piecewise constant interpolant q^Π with $q^\Pi(t) := q_{k-1}$ for $t \in [t_{k-1},t_k)$ and $q^\Pi(T) = q_M$. Choosing a sequence $(\Pi_m)_{m\in\mathbb{N}}$ of partitions, where the fineness of Π_m tends to 0 as $m\to\infty$, it is possible to apply a version of Helly's selection principle to the sequence $(q^{\Pi_m})_{m\in\mathbb{N}}$, see thereto [13]. Using (7b) and the compatibility conditions (8) it can be shown that the limit function fulfills the properties (S) and (E) of an energetic solution. For a detailed proof, see e.g. [16].

In various works this abstract theory has been applied to prove the existence of energetic solutions of rate-independent processes in the field of plasticity, damage, delamination, crack-propagation, hysteresis or phase transitions in shape memory alloys, amongst these [18, 19, 3, 20, 21, 2, 9, 10]. The way to verify the abstract conditions depends on the properties of the process under consideration. In particular, unidirectional processes such as damage or delamination processes require additional techniques to obtain compatibility condition (8b). In such a setting, the dissipation distance takes the form (2), so that it is neither continuous nor weakly continuous on $\mathcal{Z}$. Hence, (8b) cannot be directly obtained from the stability of the approximating sequence $(t_k,q_k) \rightharpoonup (t,q)$ in $[0,T]\times\mathcal{Q}$. An alternative technique to prove (8b), which is very helpful to overcome this difficulty in unidirectional problems, was introduced in [19]. It is called

Mutual recovery condition: *Let $t_j \to t$ in $[0,T]$ and $q_j \rightharpoonup q$ in $\mathcal{Q}$ with (t_j,q_j) satisfying* (S) *for all $j\in\mathbb{N}$. Then, also (t,q) satisfy* (S), *if the mutual recovery condition holds, i.e. for any $\hat q \in \mathcal{Q}$, it must be possible to construct a mutual recovery sequence $(\hat q_j)_{j\in\mathbb{N}}$ with $\hat q_j \rightharpoonup \hat q$ in $\mathcal{Q}$ so that*

$$\begin{aligned}\limsup_{j\to\infty}\,&(\mathcal{E}(t_j,\hat q_j) + \mathcal{R}(\hat z_j - z_j) - \mathcal{E}(t_j,q_j))\\ &\le \mathcal{E}(t,\hat q) + \mathcal{R}(\hat z - z) - \mathcal{E}(t,q).\end{aligned} \tag{10}$$

The term *recovery sequence* shows that this tool originates from the theory of Γ-convergence for variational problems (see

Section 4.1 and e.g. [22, 23] for more details), but *mutual* expresses that this new kind of recovery sequence has to supply more: It has to recover the limit state *mutually* in all components, *mutually* for the energy and the dissipation functional. For a mutual recovery sequence there exists no general ansatz. The way to find it is individual for each problem since the construction of the sequence strongly depends on the form of the functionals. In particular, in many applications the sequence recovering the displacement cannot be constructed independently from the one for the inner variable, since these quantities may be linked with each other by the energy functional and the dissipation potential.

3.2. Existence results for rate-independent damage

In this section we present a short overview on existence results for isotropic rate-independent damage processes and explain in which settings they hold. Examples which are well known in engineering will be discussed in Sections 3.3–3.7.

The existence of energetic solutions for the system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ with $\mathcal{E}$ from (4) and $\mathcal{R}$ from (2) was first proven in [19] for $\mathcal{G}(z)=\frac{1}{r}\int_\Omega|\nabla z|^r\,\mathrm{d}x$ with $r>d$. In [3] it was possible to extend this existence result to all $r\in(1,\infty)$. As already addressed in Section 3.1 the main difficulty lies in the proof of compatibility condition (8b), more detailed, in the construction of the mutual recovery sequence. It must guarantee that $\mathcal{R}(\hat z_j-z_j)\to\mathcal{R}(\hat z-z)$ in (10) and hence, in particular provide $\hat z_j\le z_j$ as well as $\hat z_j\rightharpoonup\hat z$ in $\mathcal{Z}$. While the construction in [19] exploits the uniform convergence of the sequence $z_j\to z$ due to the compact embedding of the Sobolev space $W^{1,r}(\Omega)$ into the space of continuous functions for $r>d$, the case $r\in(1,\infty)$ requires a completely new technique in [3]. More or less, it relies on the fact that $\min\{\hat z,z_k\}\in W^{1,r}(\Omega)$ as a combination of Sobolev functions with the Lipschitz continuous function $\min:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and utilizes that some terms of $\mathcal{G}(\min\{\hat z,z_k\})-\mathcal{G}(z_k)$ cancel in (10).

The case $r=1$ can be covered by considering the damage variables as functions of bounded variation. Then $\mathcal{G}(z)$ turns into the total variation of z. In [24] it is shown that the recovery sequence for $r\in(1,\infty)$ can be adapted to the space BV of functions of bounded variation using a theorem on the decomposability of BV-functions [25, Th. 3.84].

Due to the multiplicative link of z and e in W, such as e.g. in (5), the assumption of partial damage ($z_\star>0$) is crucial to preserve the coercivity of the energy functional with respect to e and hence, to obtain the existence of energetic solutions in terms of the displacements u. For the treatment of complete damage ($z_\star=0$) the reader is referred e.g. to [26], where it is shown that the evolution can be described in terms of energies and stresses, although the strains are no longer well-defined.

The conditions (7) and (8a) can be guaranteed under suitable assumptions on the stored energy density W, such as continuity, growth conditions on W and its derivative and coercivity. Moreover, the convexity of W with respect to the strains is required in the small strain setting [3, 24], whereas the finite strain setting demands for quasi- or polyconvex densities [19, 27].

In the subsequent Sections 3.3–3.7 we give examples on energy densities W used in engineering, which satisfy these assumptions.

3.3. Partial damage of concrete

A stored energy density which can be used to describe the damage in reinforced concrete can be considered as follows:

$$W(e,z):=\mu z|e|^2+\tfrac{\lambda}{2}\big(|(-\operatorname{tr}e)^+|^2+z|(\operatorname{tr}e)^+|^2\big)\,, \qquad (11)$$

where μ, $\lambda>0$ are the Lamé constants. As before, $e\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the linearized strain tensor and $z\in[z_\star,1]$ with $z_\star>0$ fixed is the damage variable. A special property of concrete is that damaged regions in the structure can resist compression as good as undamaged regions. This is featured in W by the fact that the term $(-\operatorname{tr}e)^+=\max\{-\operatorname{tr}e,0\}$, which stands for the volumetric part of the strains due to compression, is not affected by z. However, damaged concrete has a weaker resistance to tension. Therefore z acts on the term $(\operatorname{tr}e)^+=\max\{\operatorname{tr}e,0\}$. The assumption $z_\star>0$ models partial damage in general, but in particular, here it can be interpreted in the way that the concrete structure contains a reinforcement, which ensures that the body still reacts on tension although the concrete matrix may be completely broken. Moreover, mathematically speaking, $z_\star>0$ preserves the coercivity of W with respect to e. Since the volumetric part of the strain tensor is under control by the term $\frac{\lambda}{2}\big(|(-\operatorname{tr}e)^+|^2+z|(\operatorname{tr}e)^+|^2\big)$ it particularly ensures that also the deviatoric part is controlled. All these facts allow for finite shear stresses in the material.

It is shown in [3, Sect. 5.2] that the stored energy density W from (11) is convex and satisfies suitable growth conditions so that the corresponding energy functional fulfills the assumptions of Theorem 1. Hence, with any of the regularizations $\mathcal{G}(z)$ discussed in Section 3.2 and a complying set $\mathcal{Z}$ the existence of energetic solutions is guaranteed in the state space $\mathcal{Q}:=H^1(\Omega,\mathbb{R}^d)\times\mathcal{Z}$.

3.4. Ramberg-Osgood materials

The Ramberg-Osgood model applies to physically nonlinearly elastic materials, where the corresponding constitutive law can be described by a power law. It was introduced in [28] for aluminum alloys. Moreover, the reader is also referred to [29] for a discussion of this constitutive law. In contrast to many other materials the constitutive law for Ramberg-Osgood materials is formulated in terms of the stresses σ instead of the strains e using the complementary energy density

$$W_{cp}:\mathbb{R}^{d\times d}\to\mathbb{R}\,,\ W_{cp}(\sigma):=\tfrac12\sigma:\mathbb{A}:\sigma+\frac{a}{p'}|\sigma_{\mathrm{D}}|^{p'}\,. \qquad (12)$$

It depends on the linearized 2nd Piola-Kirchhoff stress tensor σ and its deviatoric part $\sigma_{\mathrm{D}}:=\sigma-\frac{\operatorname{tr}\sigma}{d}\mathrm{Id}$. Here $a>0$, $2<p'<\infty$ and $A\in\mathbb{R}^{(d\times d)\times(d\times d)}$ is symmetric, positive definite, i.e. $c_1^{\mathbb{A}}|\sigma|^2\le\sigma:\mathbb{A}:\sigma\le c_2^{\mathbb{A}}|\sigma|^2$ for all $\sigma\in\mathbb{R}^{d\times d}$ and constants $c_1^{\mathbb{A}},c_2^{\mathbb{A}}>0$. The complementary energy and the stored energy, the latter depending on the strains $e\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$, are linked by a Legendre transform, i.e.:

$$W(e):=\sup_{\sigma\in\mathbb{R}^{d\times d}_{\mathrm{sym}}}\big\{\sigma:e-W_{cp}(\sigma)\big\}\,.$$

In [3, Sect. 5.3] it is carried out that this energy density is convex and satisfies a coercivity estimate of form $W(e)\ge c|e|^p$ with $1/p+1/p'=1$. Moreover, it is shown for the derivative $|\partial W(e)|\le c_0W(e)+c_1$. These properties allow it to verify conditions (7) and (8a). Clearly, also the density for Ramberg-Osgood materials can be coupled with damage.

3.5. Ogden's materials at finite strains

Finite strain elasticity is a geometrically nonlinear material model. This means that also such deformations $\varphi\in\mathbb{R}^d$ can be considered, whose gradients $\nabla\varphi$ are large, so that the right Cauchy-Green tensor $C=\nabla\varphi^\top\nabla\varphi$ has to be taken into account. Therefore, one does not formulate the problem in terms of the displacement field $u(x)=\varphi(x)-x$ but directly in terms of the deformation φ and the deformation gradient $F:=\nabla\varphi$.

A physically reasonable deformation has to preserve orientation, which is ensured by

$$\nabla\varphi\in\mathrm{GL}_+(d):=\{A\in\mathbb{R}^{d\times d}\,,\,\det A>0\}\,.$$

Furthermore, the constitutive law has to be independent of rotations $R\in\mathrm{SO}(d):=\{A\in\mathbb{R}^{d\times d}\,,\,A^{-1}=A^\top,\,\det A=1\}$. This is called material frame indifference, see (13). Moreover, extreme deformations must cause extremely large values of the stored elastic energy. In particular, the interpenetration must be prevented. These two properties can be implemented in the energy density as follows:

Material frame indifference:

$$W(RF) = W(F) \quad \text{for } R \in \mathrm{SO}(d),\ F \in \mathbb{R}^{d\times d}. \tag{13}$$

Noninterpenetration:

$$\begin{cases} W(F) = +\infty & \text{for } \det F \le 0, \\ W(F) \to +\infty & \text{for } \det F \to 0_+. \end{cases} \tag{14}$$

The problem is that conditions (13) and (14) are incompatible with convexity, which is a convenient claim to ensure lower semicontinuity at small strains. To see this incompatibility, consider $P, Q \in \mathrm{SO}(d)$, $\lambda \in (0,1)$, such that $(\lambda P + (1-\lambda)Q) \notin \mathrm{SO}(d)$, which conforms to a strain. Then convexity together with material frame indifference yields the following contradiction:

$$\begin{aligned} 0 < W(\lambda P + (1-\lambda)Q) &\le \lambda W(P) + (1-\lambda)W(Q) \\ &= \lambda W(I) + (1-\lambda)W(I) = 0. \end{aligned}$$

The class of energy densities which fit to these natural requirements and which admit to prove existence are the polyconvex energy densities. They were introduced by J. M. Ball in [30] as follows: *A function $W : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ is called polyconvex if it can be represented by a function $\widetilde{W} : \mathbb{M}^d \to \mathbb{R} \cup \{\infty\}$, which is convex on the set of minors $\mathbb{M}^d$ of $\mathbb{R}^{d\times d}$-matrices.*

Ogden's materials provide a typical example for polyconvex stored energy densities, since they are defined as the sum of convex functions depending on the minors of a matrix. In three space dimensions ($d = 3$) the minors of a $F^\top F \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ are given by the trace $\mathrm{m}_1(F) := \mathrm{tr}(F^\top F)$, the trace of the cofactor matrix $\mathrm{m}_2(F) := \mathrm{tr}\,\mathrm{Cof}(F^\top F)$ and the determinant $\mathrm{m}_3(F) := \det(F^\top F)$. Hence, an Ogden's material coupled with damage can e.g. be of the form

$$W(F,z) := z\Big(\sum_{i=1}^{M} a_i \mathrm{m}_1(F)^{\frac{\gamma_i}{2}} + \sum_{i=j}^{N} b_j \mathrm{m}_2(F)^{\frac{\delta_j}{2}} + \mathrm{m}_3(F)^{-\alpha}\Big)$$

with constants $M, N \in \mathbb{N}$, $a_i, b_j > 0$, $\gamma_i, \delta_j > d$ and $\alpha > d/2$.

In [27, Sect. 3.2.3] it is carried out that the above stored energy density and its derivative satisfy suitable growth conditions, so that the corresponding energy functional fulfills (7) and (8a).

3.6. Uniformly convex density for improved temporal regularity

As it was already mentioned in Section 2, energetic solutions $(u,z) : [0,T] \to \mathcal{Q}$ may jump with respect to time. It can be shown that its temporal regularity improves under additional convexity assumptions on the energy functional $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$. In particular, in [3, Th. 4.3] it is proven that strict convexity implies that energetic solutions are continuous in time. Strict convexity of the energy functional with respect to the variables is given, if the corresponding stored energy density is jointly strictly convex in all the variables, i.e. $W(e,z)$ has to be strictly convex jointly in $(e,z) \in \mathbb{R}^{d\times d} \times [z_\star, 1]$.

It is characteristic for the modeling of damage, that the damage variable z and the strains e are linked multiplicatively by W, such as in (5), since an increase of damage usually decreases the elastic energy. The problem is that this multiplicative ansatz quite often spoils the joint convexity, although W may be separately convex in all the variables. As a negative example serves the well-known $(1-\mathsf{d})$-model for isotropic damage, see e.g. [5]: For the symmetric, positive definite fourth order tensor $\mathbb{B}$ the stored energy density

$$\hat{W}(e,\mathsf{d}) = \frac{(1-\mathsf{d})}{2} e{:}\mathbb{B}{:}e = \frac{z}{2} e{:}\mathbb{B}{:}e = W(e,z)$$

is not jointly convex in (e,z). This can be seen from calculating the Hessian; evaluating it in $(e,z){=}(e,1)$, $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, in the direction $(\tilde{e},\tilde{z}){=}(-\frac{e}{2},1)$ yields $\mathrm{D}^2W(e,z)[(\tilde{e},\tilde{z}),(\tilde{e},\tilde{z})] = z\tilde{e}{:}\mathbb{B}{:}\tilde{e} + 2\tilde{z}e{:}\mathbb{B}{:}\tilde{e} = -\frac{3}{4}e{:}\mathbb{B}{:}e < 0$.

As is shown in [3, L. 5.1], examples of jointly (strictly) convex densities can be constructed using the ansatz

$$W(e,z) := \frac{1}{\eta - z} e : \mathbb{B} : e + \frac{\alpha}{2}|z|^2 \tag{15}$$

with constants $\eta > 1$ and $\alpha \ge 0$. For $\alpha = 0$ we only have joint convexity, since then $W(0,z) = 0$ for all $z \in [z_\star, 1]$, while $\alpha > 0$ yields strict convexity jointly in (e,z).

Finally, it is worth mentioning that joint uniform convexity of W improves the temporal regularity of energetic solutions to Hölder and in some cases even to Lipschitz continuity. For more details about this result the reader is referred to [3, Sect. 4.2]. Examples can be found in [3, Sect. 5.1] or [31, Sect. 5.3.2]. Since the density in (15) is of quadratic growth in e, it is even uniformly convex. Hence, using a quadratic regularization for ∇z will lead to an energy functional which is uniformly convex on $\mathcal{Q} := H^1(\Omega,\mathbb{R}^d) \times H^1(\Omega)$. Thus, energetic solutions of this problem are Lipschitz continuous with respect to time. The Hölder continuity of energetic solutions plays a role if either the stored energy density or the regularization is of superquadratic growth with respect to e or ∇z, whereas subquadratic growth still leads to Lipschitz continuity.

3.7. Bang-bang damage

As an example for the model with BV-regularization we want to mention the so-called bang-bang damage model, which is discussed in [24]. It describes the special case, when the damage variable attains the values 1 or 0, only. This means that the damage variable $z : \Omega \to \{0,1\}$ only distinguishes between the two situations: locally unbroken for $z(x) = 1$ and locally broken for $z(x) = 0$. In this setting, the set of admissible damage variables can be considered to be the subset of $\mathrm{BV}(\Omega)$ consisting of the characteristic functions of sets of finite perimeter, i.e.

$$\mathcal{Z}_{BB} := \left\{ \begin{array}{r} \chi_Z : \Omega \to \{0,1\} \text{ characteristic function of} \\ Z \subset \Omega, P(Z,\Omega) < \infty \end{array} \right\}. \tag{16}$$

Since a characteristic function χ_Z of such a set Z is simply a jump function, its variation in Ω reduces to the jump part, which here is equivalent to the perimeter of Z in Ω, i.e. $|\mathrm{D}\chi_Z|(\Omega) = P(Z,\Omega)$. Hence, the regularization for the damage variable $z = \chi_Z$ in this setting is $\mathcal{G}_{BB}(z) := \sigma P(Z,\Omega)$ with a constant $\sigma > 0$. Compactness properties of the set $\mathcal{Z}_{BB}$ and properties of $\mathcal{G}_{BB}$ used for bang-bang damage are explicated in [24, Sect. 3] and we also refer to [25, Chap. 4] for more details.

4. Delamination models

In many engineering contributions (e.g. in [32]) an interface is understood as the limit of a thin medium following its own constitutive law. This was the motivation to mathematically rigorously perform such a limit passage in order to obtain a delamination model as the limit of damage models. [33] considers a three-specimen-sandwich-structure, where the outer two constituents Ω_-, Ω_+ are perfectly unbreakable and the middle component $\Omega^\varepsilon_{\mathrm{D}}$ experiences partial damage. Here, the lower bound on the damage variable depends on a parameter $\varepsilon > 0$, i.e. $z^\varepsilon_\star = \varepsilon^\gamma \in (0,1)$ for a positive γ, such that $z(t,x) \in [\varepsilon^\gamma, 1]$ for a.e. $(t,x) \in [0,T] \times \Omega$. As $\varepsilon \to 0$ the lower bound $z^\varepsilon_\star$ as well as the thickness 2ε of the damageable component $\Omega^\varepsilon_{\mathrm{D}}$ tend to 0, so that the limit model describes delamination along the interface Γ_{C}, since z may attain the value 0 in the limit. The limit passage is done in a double limit. We start in the setting described above with an energy functional $\mathcal{E}^\kappa_\varepsilon$ similar to (4), but defined on the domain $\Omega = \Omega_- \cup \Omega^\varepsilon_{\mathrm{D}} \cup \Omega_+$ and containing the regularization $\mathcal{G}^\kappa_\varepsilon(z) = \int_{\Omega^\varepsilon_{\mathrm{D}}} \frac{\kappa}{\varepsilon}|\nabla z|^r\,\mathrm{d}x$, $r \in (1,\infty)$. Moreover, we use the modified dissipation potential $\mathcal{R}_\varepsilon(v) = \frac{1}{\varepsilon}\mathcal{R}(v)$. Then, the first limit $\varepsilon \to 0$ leads to a model involving the delamination gradient, which is called gradient delamination model:

$$\mathcal{E}^\kappa(t,u,z) = \begin{cases} \int\limits_{\Omega_-\cup\Omega_+} W(e(u))\mathrm{d}x - \langle l(t),u\rangle + \mathcal{G}^\kappa(z) & \text{if } (u,z)\in\mathcal{Q}_\mathrm{C}, \\ \infty & \text{otherwise}, \end{cases} \quad (17)$$

where $\mathcal{G}^\kappa(z) = \int_{\Gamma_\mathrm{C}} 2\kappa|\nabla z|^r\,\mathrm{d}x$. Because of this, the delamination variable can attain values between 0 and 1. This property differs from those of crack-models based on Griffith' fracture criterion [34], as studied e.g. in [35, 36, 37]. To overcome this discrepancy the gradient is suppressed in a second limit $\kappa \to 0$:

$$\mathcal{E}(t,u,z) = \begin{cases} \int\limits_{\Omega_-\cup\Omega_+} W(e(u))\mathrm{d}x - \langle l(t),u\rangle & \text{if } (u,z)\in\mathcal{Q}_\mathrm{G}, \\ \infty & \text{otherwise}. \end{cases} \quad (18)$$

In fact, this second limit model is of Griffith-type, since it is shown in [33] that an energetic solution $(u,z):[0,T]\to\mathcal{Q}$ satisfies for all $t\in[0,T]$ that $z(t,x)\in\{1,0\}$ for a.e. $x\in\Gamma_\mathrm{C}$ and the initial datum $z_0 = 1$. Both models incorporate transmission and unilateral contact conditions along the interface:

$$z[\![u]\!] = 0 \quad \text{and} \quad [\![u\cdot \mathrm{n}_1]\!] \geq 0 \quad \text{a.e. on } \Gamma_\mathrm{C}\,, \quad (19)$$

which are reflected in (17) and (18) by the sets $\mathcal{Q}_\mathrm{C}$ and $\mathcal{Q}_\mathrm{G}$ and which result from the following ansatz for the stored energy density on $\Omega_\mathrm{D}^\varepsilon$ in the damage model:

$$W_\mathrm{D}(e,z) := zW(e) + |\max\{-\operatorname{tr} e, 0\}|^p\,. \quad (20)$$

Here $W:\mathbb{R}^{d\times d}_\mathrm{sym}\to\mathbb{R}$ is a convex, coercive stored energy density, $[\![u]\!]$ denotes the jump of u across Γ_C and n_1 is the unit normal vector to Γ_C. Obtaining the Griffith-type delamination model, i.e. passing to 0 with κ in (17), requires continuity of the displacements. Because of the compact embedding of the Sobolev space $W^{1,p}(\Omega,\mathbb{R}^d)$ for $p>d$ into the space of continuous functions, this is ensured if W is of superquadratic growth in e, more detailed if $c_1|e|^p \leq W(e) \leq c_2|e|^p$ with $p>d$.

4.1. *Tools: Γ-convergence of rate-independent systems*

The mathematical tool which is used to perform the above limit passages $\varepsilon \to 0$ and $\kappa \to 0$ is the Γ-convergence of rate-independent systems. It was developed in [16] by adapting the original notion of Γ-convergence of functionals to rate-independent processes.

However Γ-convergence introduced by De Giorgi in [22], see also [23] is a method to gain a static variational problem and its minimizers as a limit of a sequence of static variational problems and their minimizers. A sequence of functionals $(\mathcal{F}_j)_{j\in\mathbb{N}}$ with $\mathcal{F}_j:\mathcal{X}\to\mathbb{R}\cup\{\infty\}$, where $\mathcal{X}$ is a metric space, is said to Γ-converge to a functional $\mathcal{F}:\mathcal{X}\to\mathbb{R}\cup\{\infty\}$ if for every $w\in\mathcal{X}$ the following two conditions are satisfied:

Γ- lim inf inequality:

$$\forall\,(w_j)_{j\in\mathbb{N}},\; w_j \overset{\mathcal{X}}{\to} w:\; \mathcal{F}(w) \leq \liminf_{j\to\infty}\mathcal{F}_j(w_j)\,, \quad (21a)$$

Recovery sequence:

$$\exists\,(\hat{w}_j)_{j\in\mathbb{N}},\; \hat{w}_j \overset{\mathcal{X}}{\to} w:\; \mathcal{F}(w) \geq \limsup_{j\to\infty}\mathcal{F}_j(w_j)\,. \quad (21b)$$

For rate-independent systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)_{j\in\mathbb{N}}$ with time-dependent, energetic solutions $q_j:[0,T]\to\mathcal{Q}$ it is easy to see that

$$\mathcal{E}_\infty = \Gamma\text{-}\lim\mathcal{E}_j \quad \text{and} \quad \mathcal{R}_\infty = \Gamma\text{-}\lim\mathcal{R}_j$$

is not sufficient to ensure (S) and (E). Hence a modified concept of Γ-convergence has to be used for the quasistatic setting and the energetic formulation of rate-independent processes. It is desired that energetic solutions $q_j:[0,T]\to\mathcal{Q}$ of the approximating systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)$ converge to an energetic solution $q:[0,T]\to\mathcal{Q}$ of the limit system $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{R}_\infty)$. This means that conditions (3) must be maintained under convergence, so that the interplay of $\mathcal{E}_j$ and $\mathcal{R}_j$ is important. In [16] the theory of Γ-convergence was adapted to the framework of the energetic formulation of rate-independent processes. In particular [16, Th. 3.1] supplies sufficient conditions which guarantee that a subsequence of energetic solutions of the approximating systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)$ converges to an energetic solution of the limit system $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{R}_\infty)$.

One of these sufficient conditions is the mutual recovery condition which can be adapted from (10) to the above setting of sequences of functionals by replacing the functionals $\mathcal{E}$ and $\mathcal{R}$ on the left-hand side of the inequality (10) by the approximating functionals $\mathcal{E}_j$ and $\mathcal{R}_j$, respectively. It should be mentioned that the construction of the mutual recovery sequences for the limit passages $\varepsilon\to 0$ and $\kappa\to 0$ are extraordinarily difficult, since the transmission condition (19) induces a very strong link between the damage variable and the displacement. In order to recover it in the limit, this link has to be implemented into mutual recovery sequence by a suitable adaption to the state spaces $\mathcal{Q}_\mathrm{D}$ and $\mathcal{Q}_\mathrm{C}$ of the approximating problems. For more details we refer the reader to [33, Sect. 3.2 and 4.2].

4.2. *Alternative model: Adhesive delamination*

The Griffith-type delamination model $(\mathcal{Q},\mathcal{E},\mathcal{R})$ may be disadvantageous to implement numerically due to the local nature of the transmission condition and the noninterpenetration condition (19). As it shown in [33, Sect. 5] the double limit passage can also be performed simultaneously by choosing ε suitably in dependence of κ. Hence, the local constraints (19) can be compassed by using the partial damage model $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R}_\varepsilon)$ for sufficiently small $\varepsilon(\kappa)$ and κ.

An alternative ansatz is suggested in [20]. Using the theory of Γ-convergence of rate-independent systems it is shown that the local transmission condition $z[\![u]\!] = 0$ can be approximated by the nonlocal energy term

$$\int_{\Gamma_\mathrm{C}} kz\big|[\![u]\!]\big|^2\,\mathrm{d}x \qquad \text{as } k\to\infty\,.$$

The corresponding rate-independent model is the so-called adhesive delamination model. This notion is due to the fact that the displacement are allowed to jump across Γ_C even for $z>0$. This reflects that $z:\Gamma_\mathrm{C}\to[z_\star,1]$ describes the state of an adhesive which bonds the separate bodies Ω_- and Ω_+ along Γ_C.

5. Summary and concluding remarks

This contribution presents damage and delamination models which are stated using the energetic formulation of rate-independent processes. The suitable notion of solution in this context is the so-called energetic solution. It is defined by (S) & (E). An abstract existence result is given by Theorem 1 in Section 3.1. It is addressed how this result is used to prove the existence of energetic solutions for the model for partial isotropic damage presented in Section 3 with different regularization terms for the damage variable. Section 4 discusses several delamination models as the limit of damage models. These are the gradient delamination model and the Griffith-type delamination model. An alternative model is the adhesive delamination model described in Section 4.2. The mathematical tool for this limit passage is the Γ-convergence of rate-independent systems, which is introduced in Section 4.1. The Sections 3.3–3.7 discuss examples of stored energy densities coupled with damage which fit into the framework of Theorem 1 and which are relevant in engineering, such as the damage of reinforced concrete and of Ramberg-Osgood or

Ogden's materials. Moreover, Section 3.6 introduces an alternative ansatz for the multiplicative coupling of the damage variable and the strains which leads to improved temporal regularity of energetic solutions.

Section 3.7 discusses the bang-bang model, where the damage variable is a characteristic function of a set Z with finite perimeter and the regularization is $\mathcal{G}_{BB}(z) = \sigma P(Z,\Omega)$ is given by the perimeter. However, for numerical simulations it may be unfavorable to work with a regularization which is a measure. For this purpose one can approximate $\mathcal{G}_{BB}$ by the regularizations

$$\mathcal{G}_k(z) := \begin{cases} \int_\Omega \left(k^2(z-\varepsilon_k)^2(1-z)^2 + \frac{1}{k^2}|\nabla z|^2\right) \mathrm{d}x & \\ & \text{if } z \in H^1(\Omega,[\varepsilon_k,1])\,, \\ \infty & \text{otherwise,} \end{cases}$$

where $\varepsilon_k \to 0$ as $k \to \infty$ and $\sigma := 2\int_0^1 z(1-z)\,\mathrm{d}z$. A term of the form $\mathcal{G}_k$ was first used in [38] for the modeling of phase separation. In a static context it was shown that the functionals $\mathcal{G}_k$ Γ-converge to $\mathcal{G}_{BB}$, see also [39]. In [24, Sect. 3.1] it is verified that this approximation ansatz can be carried over to the rate-independent setting for the modeling of partial isotropic damage, where the unidirectionality of the dissipation potential imposes an additional constraint. At this point it is worth mentioning that a regularization of the form $\mathcal{G}_k$ also occurs in the so-called Ambrosio-Tortorelli model for rate-independent damage. It is proved in [37] that these models approximate the Francfort-Marigo model for Griffith-type fracture, see [12]. This result is again based on the Γ-convergence of rate-independent processes.

In this work damage and delamination phenomena are modeled as rate-independent, unidirectional processes. This modeling assumption is admissible whenever the external loadings evolve much slower than the internal (damage or delamination) variable and if the healing of the material is impossible. This is true for metals, concrete or rocks. But as soon as one wishes to model the damage of living structures such as bones, these assumptions fail. Fortunately, healing of microcracks in bones is possible and it is a process which is performed steadily by very complex mechanisms. The healing of the bone is slower than the external loadings acting on it and possibly much slower than its damage, which can happen instantaneously. Therefore, the mathematical modeling of bones goes beyond rate-independence and it entails many interesting questions and analytical challenges.

References

[1] Kachanov, L. M., *Introduction to Continuum Damage Mechanics*, Kluwer Academic Publishers, 1990.

[2] Mielke, A., Theil, F., On rate-independent hysteresis models, NoDEA 11(2), pp. 151-189, 2004.

[3] Mielke, A., Thomas, M., Damage in nonlinearly elastic materials at small strain – existence and regularity results, *ZAMM*, 90(2), pp. 88-112, 2010.

[4] Frémond, M., Nedjar, B., Damage, gradient of damage and the principle of virtual power, *Int. J. Solids Struct.*, 33, pp. 1083-1103, 1996.

[5] Lemaitre, J., Desmorat, R., *Engineering damage mechanics*, Springer, Berlin, Heidelberg, New York, 2005.

[6] Hackl, K., Stumpf, H., Micromechanical concept for the analysis of damage evolution in thermo-viscoelastic and quasi-static brittle fracture, *Int. J. Solids Struct.*, 30, pp. 1567-1584, 2003.

[7] Lorentz, E., Benallal, A., Gradient constitutive relations: Numerical aspects and applications to gradient damage, *Comput. Methods Appl. Mech. Engrg.*, 194, pp. 5191-5220, 2005.

[8] Dal Maso, G., De Simone, A., Mora, M. G., Quasistatic evolution problems for linearly elastic - perfectly plastic materials, *Arch. Ration. Mech. Anal.*, 180, pp. 237-291, 2006.

[9] Mielke, A., Petrov, A., Thermally driven phase transformation in shape-memory alloys, *Gakk ōtosho (Adv. Math. Sci. Appl.)*, 17, pp. 667-685, 2007.

[10] Mielke, A., Paoli, L., Petrov, A., On existence and approximation for a 3D model of thermally-induced phase transformations in shape-memory alloys, *SIAM J. Math. Anal.*, 41, pp. 1388-1414, 2009.

[11] Dal Maso, G. and Francfort, G. and Toader, R., Quasistatic crack growth in nonlinear elasticity, *Arch. Ration. Mech. Anal.*, 176(2), pp. 165-225, 2004

[12] Bourdin, B. and Francfort, G. and Marigo, J.-J., The Variational Approach to Fracture, *J. Elasticity*, 91, pp. 5-148, 2008.

[13] Mainik, A., Mielke, A., Existence results for energetic models for rate-independent systems, *Calc. Var. PDEs*, 22, pp. 73-99, 2005.

[14] Mielke A., Evolution in rate-independent systems (Ch. 6), *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, eds.: Dafermos, C.M., Feireisl, E., Elsevier B.V., pp. 461-559, 2005.

[15] Francfort, G., Mielke, A., Existence results for a class of rate-independent material models with nonconvex elastic energies, *J. reine angew. Math.*, 595, pp. 55-91, 2006.

[16] Mielke, A., Roubíček, T., Stefanelli, U., Γ-limits and relaxations for rate-independent evolutionary problems, *Calc. Var. PDEs*, 31, pp. 387-416, 2008.

[17] Mielke, A., Differential, energetic and metric formulations for rate-independent processes, *C.I.M.E. Summer School on Nonlinear PDEs and Applications, Cetraro & WIAS-Preprint* 1454, 2009.

[18] Mainik, A., Mielke, A., Global existence for rate-independent gradient plasticity at finite strain, *J. Nonlinear Science*, 19(3), pp. 221-248, 2009.

[19] Mielke, A., Roubíček, T., Rate-independent damage processes in nonlinear elasticity, *M^3AS*, 16, pp. 177-209, 2006.

[20] Roubíček, T., Scardia, L., Zanini, C., Quasistatic delamination problem, *Cont. Mech. Thermodynam.*, 21, pp. 223-235, 2009.

[21] Knees, D., Mielke, A., Zanini, C., On the inviscid limit of a model for crack propagation, *M^3AS*, 18(9), pp. 1529-1569, 2008.

[22] De Giorgi, E., Γ-convergenza e G-convergenza, *Boll. Unione Math. Ital., V. Ser.* A 14, pp. 213-220, 1977.

[23] Dal Maso, G., *An Introduction to Γ-Convergence*, Birkhäuser Boston Inc., Boston, MA, 1993.

[24] Thomas, M., Rate-independent damage evolution with BV-regularization, *Rate-independent evolutions*, eds.: Dal Maso, G., Mielke, A., Stefanelli, U., DCDS-S, to appear in 2011.

[25] Ambrosio, L.,Fusco, N., Pallara, D., *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.

[26] Mielke, A., Complete-damage evolution based on energies and stresses, *Special Issue of Discrete Cont. Dyn. Syst. Ser. S* 4, pp. 423-439, 2011.

[27] Thomas, M., *Rate-independent damage processes in nonlinearly elastic materials*, PhD-thesis, Humboldt-Universität zu Berlin, 2010.

[28] Osgood, W.R., Ramberg, W., Description of stress-strain curves by three parameters. NACA Technical Note 902, *National Bureau of Standards, Washington*, 1943.

[29] Knees, D., *Regularity results for quasilinear elliptic systems of power-law growth in nonsmooth domains – Boundary, transmission and crack problems*, PhD-thesis, Stuttgart University, 2005.

[30] Ball, J.M., Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4), pp. 337-403, 1976.

[31] Gruber, P., Knees, D., Nesenenko, S., Thomas, M., Analytical and numerical aspects of time-dependent models with internal variables. *ZAMM*, 90, pp. 861-902, 2010.

[32] Allix, O., Interface Damage Mechanics: Application to Delamination, *Continuum Damage Mechanics of Materials and Structures*, eds.: Allix, O., Hild, F., Elsevier, pp. 295-325, 2002.

[33] Mielke, A., Roubíček, T., Thomas, M., From damage to delamination in nonlinearly elastic materials at small strains, *WIAS-Preprint* 1542, submitted to J. of Elasticity.

[34] Griffith, A.A., The Phenomena of Rupture and Flow in Solids, *Phil. Trans. R. Soc. Lond. A* 221, pp. 163-198, Jan. 1921.

[35] Dal Maso, G., Toader, R., A Model for the Quasi-Static Growth of Brittle Fractures: Existence and Approximation Results, *Arch. Ration. Mech. Anal.*, 162(2), pp. 101-135, 2002.

[36] Francfort, G., Larsen, C., Existence and Convergence for Quasi-Static Evolution in Brittle Fracture, *Comm. Pure Appl. Math.*, 56(10), pp. 1465-1500, 2003.

[37] Giacomini, A., Ambrosio-Tortorelli approximation for quasi-static evolution of brittle fractures, *Calc. Var. PDEs*, 22, pp. 129-172, 2005.

[38] Modica, L., Mortola, S., Un esempio di Γ-convergenza, *Boll. U. Mat. Ital. B*, 14(5), pp. 285-299, 1977.

[39] Alberti, G., Variational Models for Phase Transitions, an Approach via Gamma-Convergence, *Differential Equations and Calculus of Variations*, eds.: Buttazzo, G., et al., Springer, 2000.