Shape sensitivity analysis of time-dependent flows of incompressible fluids

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Abstract

We consider a flow of an incompressible fluid around an obstacle whose shape is to be optimized subject to the work functional. The expressions for the material derivative of the solution to the flow problem and the shape derivative of the functional are shown and rigorously justified.

Keywords: fluid mechanics, shape design, sensitivity

1. Introduction

We consider the flow of an incompressible fluid in a bounded domain $\Omega := B \setminus S \subset \mathbb{R}^d$ of the class $\mathcal{C}^{1,1}$, where B is a container, S is an obstacle whose shape is to be optimized and $d \in \{2, 3\}$.

Motion of the fluid is described by the system of equations

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbb{S}(\mathbb{D}(\boldsymbol{v})) + \nabla p + \mathbb{C}\boldsymbol{v} = \boldsymbol{f}, \tag{1a}$$

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1b}$$

in $Q_T := (0, T) \times \Omega$, supplemented with the Navier slip boundary condition

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad (\mathbb{T}\boldsymbol{n})_\tau = -a\boldsymbol{v}_\tau \tag{2}$$

on $\Sigma_T := (0, T) \times \partial\Omega$ and the intial condition

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \text{ in } \Omega. \tag{3}$$

Here $\mathbb{T} = -p\mathbb{I}+\mathbb{S}$ is the Cauchy stress tensor whose traceless part $\mathbb{S} := \mathbb{S}(\mathbb{D}(\boldsymbol{v}))$ satisfies certain growth conditions, for simplicity of notations we restrict ourselves to the power-law model

$$\mathbb{S}(\mathbb{D}(\boldsymbol{v})) = (1 + |\mathbb{D}(\boldsymbol{v})|)^{r-2}\mathbb{D}(\boldsymbol{v}). \tag{4}$$

The symbol $\mathbb{D}(\boldsymbol{v})$ stands for the symmetrized velocity gradient, $\mathbb{C}$ is a skew-symmetric Coriolis term, $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$. The condition (2) describes an impermeable surface along which the fluid can slip with certain friction. The corresponding friction coefficient a is assumed to be positive, for simplicity we set $a = 1$.

We want to minimize or maximize (depending on the physical motivation) the work functional, which for the specific problem is given by

$$J(\Omega) := -\int_0^T \int_{\partial S} \mathbb{T}\boldsymbol{n} \cdot \boldsymbol{v} = \int_0^T \int_{\partial S} |\boldsymbol{v}|^2. \tag{5}$$

For this reason we develop the expression for the shape derivative of J which will allow us to apply a gradient based algorithm for its minimization.

If $r \geq (d + 2)/2$, $T > 0$ and $\Omega \in \mathcal{C}^{1,1}$ then the problem (1)–(3) has a unique weak solution $(\boldsymbol{v}, p) \in \left[L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^r(0, T; \mathbf{W}^{1,r}_N(\Omega))\right] \times L^{r'}(0, T; L^{r'}_0(\Omega))$, where $r' := r/(r - 1)$ and

$$\mathbf{W}^{1,r}_N(\Omega) := \{\boldsymbol{\phi} \in \mathbf{W}^{1,r};\ \operatorname{tr} \boldsymbol{\phi} \cdot \boldsymbol{n} = 0\}, \tag{6}$$

see e.g. [1] for the existence theory. In what follows we will assume that the above conditions are satisfied.

2. Formulation in the fixed domain

We choose a vector field $\boldsymbol{T} \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ vanishing in the vicinity of ∂B and define the mapping

$$\boldsymbol{y} = \boldsymbol{x} + \varepsilon\boldsymbol{T}(\boldsymbol{x}), \tag{7}$$

which describes the perturbation of the boundary ∂S. For small $\varepsilon > 0$ the mapping $\boldsymbol{x} \mapsto \boldsymbol{y}$ takes diffeomorphically the region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$ where $S_\varepsilon = \boldsymbol{y}(S)$.

Let $(\bar{\boldsymbol{v}}_\varepsilon, \bar{p}_\varepsilon)$ be the solution of problem (1)–(3) on $(0, T) \times \Omega_\varepsilon$. Introducing the transformations

$$\boldsymbol{v}_\varepsilon(t, \boldsymbol{x}) := \mathbb{N}^\top(\boldsymbol{x})\bar{\boldsymbol{v}}_\varepsilon(t, \boldsymbol{y}(\boldsymbol{x})), \quad p_\varepsilon(t, \boldsymbol{x}) := \bar{p}_\varepsilon(t, \boldsymbol{y}(\boldsymbol{x})), \tag{8}$$

where

$$\mathbb{N} := \mathfrak{g}\mathbb{M}^{-1},\ \mathbb{M} := \mathbb{I} + \varepsilon\, \mathrm{D}\, \boldsymbol{T},\ \mathfrak{g} := \det \mathbb{M}, \tag{9}$$

one can show that the new pair $(\boldsymbol{v}_\varepsilon, p_\varepsilon)$ is the weak solution of the problem

$$\partial_t \boldsymbol{v}_\varepsilon + \operatorname{div}(\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \operatorname{div} \mathbb{S}(\mathbb{D}(\boldsymbol{v}_\varepsilon)) + \nabla p_\varepsilon + \mathbb{C}\boldsymbol{v}_\varepsilon = \boldsymbol{f} + \boldsymbol{A}_\varepsilon, \tag{10a}$$

$$\operatorname{div} \boldsymbol{v}_\varepsilon = 0 \tag{10b}$$

in the fixed domain Q_T with the same boundary conditions, where $\boldsymbol{A}_\varepsilon \in \left[L^r(0, T; \mathbf{W}^{1,r}_N(\Omega))\right]^*$ is a term of order ε.

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3. Shape stability of weak solutions

Before studying the shape differentiability we need to show the following stability result.

Proposition 1 *Let $r \geq (d+2)/2$. Then there is a constant $C > 0$ such that for sufficiently small $\varepsilon \geq 0$:*

$$\sup_{t\in(0,T)} \|\boldsymbol{v}_\varepsilon(t)\|_2^2 + \int_0^T \Big(\|\boldsymbol{v}_\varepsilon\|_{1,r}^r + \|\boldsymbol{v}_\varepsilon\|_{2,\partial\Omega}^2 + \|\partial_t \boldsymbol{v}_\varepsilon\|_{\mathbf{W}_N^{-1,r'}(\Omega)}^{r'} + \|p_\varepsilon\|_{r'}^{r'} \Big) \leq C. \quad (11)$$

The whole sequence $\{(\boldsymbol{v}_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ tends to $(\boldsymbol{v}, p)$ in the following sense:

$$\begin{aligned}
\boldsymbol{v}_\varepsilon &\rightharpoonup^* \boldsymbol{v} && \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
\boldsymbol{v}_\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^r(0,T;\mathbf{W}_N^{1,r}(\Omega)) \cap \mathbf{L}^{z(r)}(Q_T)\\
& && \text{and in } \mathbf{L}^2(\Sigma_T),\\
\partial_t \boldsymbol{v}_\varepsilon &\rightharpoonup \partial_t \boldsymbol{v} && \text{in } L^{r'}(0,T;\mathbf{W}_N^{-1,r'}(\Omega)),\\
p_\varepsilon &\rightharpoonup p && \text{in } L^{r'}(Q_T),\ \varepsilon \to 0.
\end{aligned}$$

Here $z(r) := r\frac{d+2}{d}$.

Note that the stability can be proved for a larger class of fluids in terms of the power-law index r, see [4].

4. Existence of material derivatives

We are going to estimate the differences

$$(\boldsymbol{u}_\varepsilon, q_\varepsilon) := (\frac{\boldsymbol{v}_\varepsilon - \boldsymbol{v}}{\varepsilon}, \frac{p_\varepsilon - p}{\varepsilon}) \quad (13)$$

and identify their limit as the solution to a linearized problem.

Proposition 2 *Let $r \geq (d+2)/2$. Then there is a constant $C > 0$ independent of $\varepsilon > 0$ such that*

$$\sup_{t\in(0,T)} \|\boldsymbol{u}_\varepsilon(t)\|_2^2 + \|\mathbb{D}(\boldsymbol{u}_\varepsilon)\|_{2,Q_T}^2 + \|\boldsymbol{u}_\varepsilon\|_{2,\Sigma_T}^2 \leq C. \quad (14)$$

If $(d+2)/2 \leq r < 4$ then we additionally have:

$$\int_0^T \|q_\varepsilon\|_{\frac{2r}{3r-4}}^{\frac{2r}{3r-4}} + \int_0^T \|\partial_t \boldsymbol{u}_\varepsilon\|_{\mathbf{W}_N^{-1,\frac{2r}{3r-4}}(\Omega)}^{\frac{2r}{3r-4}} \leq C. \quad (15)$$

In order to guarantee that the sequence $\{(\boldsymbol{u}_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ has a (unique) limit, we require an additional assumption on the regularity of solutions, namely that

$$\boldsymbol{v}_\varepsilon, \boldsymbol{v} \in L^\infty(0,T;\mathbf{W}_N^{1,\infty}(\Omega)) \cap \mathbf{W}^{1,2}(0,T;\mathbf{W}_N^{-1,2}(\Omega)) \quad \text{(R)}$$

uniformly w.r.t. $\varepsilon > 0$. The assumption (R) can be guaranteed in terms of the data at least in the case $d = 2$, see e.g. [2, 3].

Theorem 3 *Let the assumptions of the previous proposition be satisfied and* (R) *hold. Then*

$$\begin{aligned}
\boldsymbol{u}_\varepsilon &\rightharpoonup^* \dot{\boldsymbol{v}} && \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
\boldsymbol{u}_\varepsilon &\rightharpoonup \dot{\boldsymbol{v}} && \text{in } L^2(0,T;\mathbf{W}_N^{1,2}(\Omega)),\\
\boldsymbol{u}_\varepsilon &\to \dot{\boldsymbol{v}} && \text{strongly in } \mathbf{L}^{z(2)}(Q_T) \text{ and in } \mathbf{L}^2(\Sigma_T),\\
q_\varepsilon &\rightharpoonup \dot{p} && \text{in } L^2(Q_T),\\
\partial_t \boldsymbol{u}_\varepsilon &\rightharpoonup \partial_t \dot{\boldsymbol{v}} && \text{in } L^2(0,T;\mathbf{W}_N^{-1,2}(\Omega)),\\
\frac{\boldsymbol{A}_\varepsilon}{\varepsilon} &\rightharpoonup \boldsymbol{A}_0' && \text{in } L^2(0,T;\mathbf{W}_N^{-1,2}(\Omega)),
\end{aligned}$$

where $(\dot{\boldsymbol{v}}, \dot{p})$ is the material derivative of $(\boldsymbol{v}, p)$ and the unique weak solution to the following problem:

$$\partial_t \dot{\boldsymbol{v}} + \operatorname{div}(\dot{\boldsymbol{v}} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \dot{\boldsymbol{v}}) - \operatorname{div}(\mathbb{S}'(\mathbb{D}(\boldsymbol{v}))\mathbb{D}(\dot{\boldsymbol{v}})) + \nabla \dot{p} + \mathbb{C}\dot{\boldsymbol{v}} = \boldsymbol{A}_0', \quad (16)$$

$$\operatorname{div} \dot{\boldsymbol{v}} = 0 \quad (17)$$

in Q_T, completed by the boundary and initial conditions

$$\dot{\boldsymbol{v}} \cdot \boldsymbol{n} = 0, \quad \big[\mathbb{S}'(\mathbb{D}(\boldsymbol{v}))\mathbb{D}(\dot{\boldsymbol{v}})\boldsymbol{n}\big]_\tau = -\dot{\boldsymbol{v}}_\tau \text{ on } \Sigma_T, \quad (18)$$

$$\dot{\boldsymbol{v}}(0,\cdot) = \dot{\boldsymbol{v}}_0. \quad (19)$$

5. Shape derivative of the cost function

Since $\boldsymbol{T} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^d)$, one can easily check that the matrix $\mathbb{N}$ admits the expansion:

$$\mathbb{N} = \mathbb{I} + \varepsilon \mathbb{N}' + O(\varepsilon^2), \quad (20)$$

in $\mathcal{C}^1(\overline{\Omega})$, where

$$\mathbb{N}' := \frac{\mathrm{d}\mathbb{N}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = (\operatorname{div} \boldsymbol{T})\mathbb{I} - \mathrm{D}\,\boldsymbol{T}. \quad (21)$$

Theorem 4 *The shape derivative of the cost function is given by*

$$\frac{\mathrm{d}J}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = J_e(\boldsymbol{T}) + J_{\boldsymbol{v}}(\dot{\boldsymbol{v}}), \quad (22)$$

where

$$J_e(\boldsymbol{T}) := \int_0^T \int_{\partial S} ((\boldsymbol{n} \cdot \mathbb{N}'\boldsymbol{n})|\boldsymbol{v}|^2 - 2\mathbb{N}'\boldsymbol{v} \cdot \boldsymbol{v}), \quad (23)$$

$$J_{\boldsymbol{v}}(\dot{\boldsymbol{v}}) := 2\int_0^T \int_{\partial S} \boldsymbol{v} \cdot \dot{\boldsymbol{v}}. \quad (24)$$

6. Conclusion and acknowledgement

The paper deals with the obstacle problem for time-dependent flow of an incompressible fluid with the viscosity depending on the shear rate. We have shown that the respective work functional is differentiable with respect to the perturbation of the shape of the obstacle and derived the expression for the shape gradient.

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