Optimal Control Problems for a Class of Evolution Inclusions with Applications to Dynamic and Quasi-static Viscoelastic Contact Problems

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Abstract

In the paper we study the optimal control problems for systems described by nonlinear evolution inclusions. Two classes of evolution inclusions of second order (hyperbolic) and of first order (parabolic) are considered. We provide results on the existence of weak solution to the evolution inclusions and of optimal solutions for the control problems. Then we deal with dynamic and quasi-static viscoelastic contact problems with nonmonotone normal compliance and friction. For these contact problems we formulate control problems. We show that the abstract results are applicable to evolution and quasi-static nonlinear hemivariational inequalities which model contact problems. Finally, we also provide a result on the variational stability of optimal control problems when the acceleration in the system is negligible.

Keywords: hemivariational inequality, quasi-static, vanishing acceleration, variational stability, Clarke subdifferential, weak solution, hyperbolic, parabolic, optimal solution

1. Introduction

In this paper we study the mechanical problems modeled by an abstract second order nonlinear evolution hemivariational inequality of the form

$$u''(t) + A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni f(t) \tag{1}$$

for a.e. $t \in (0, T)$, where a family of operators $A(t)$ and B are linear continuous operators from a reflexive Banach space V into its dual V^*, M is a linear continuous operator, M^* is its adjoint, ∂J denotes the Clarke subdifferential of a locally Lipschitz time-dependent function $J(t,\cdot)$ and $f\colon (0,T) \to V^*$ is prescribed. Moreover, appropriate initial conditions are added to (1).

The investigation of this type of inclusions is motivated by several contact problems of solid mechanics. It is well known that the dynamic equations of motion, representing momentum conservation, that govern the evolution of the state of the body, are of the form $u_i'' - \sigma_{ij,j} = f_i$, where u is a displacement, σ is the stress tensor and f is the density (per unit volume) of applied forces, such as gravity. These equations are valid for all systems and materials, since they are derived from the fundamental principle of momentum conservation (cf. e.g. [5, 6, 16]).

We also associate with model (1) the following parabolic problem which is obtained from (1) in the limit as the mass coefficient in the inertial term tends to zero

$$A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni f(t) \tag{2}$$

for a.e. $t \in (0, T)$. The hemivariational inequalities were considered for the first time in the 1980s by P.D. Panagiotopoulos [14, 15, 13] who introduced this notion as a generalization of variational inequality. The dynamic hemivariational inequalities model nonmonotone and nonsmooth subdifferential boundary conditions in contact mechanics, cf. e.g. [2, 7, 8, 9, 10, 12]. On the other hand, the quasistatic hemivariational inequalities of type (2) have been studied in the literature only recently, cf. [11]. The hemivariational inequalities are based on a notion of the Clarke subdifferential for locally Lipschitz functions, cf. [1, 3, 4].

The goal of the paper is to study the optimization and control problems for both (1) and (2). We prove the existence of optimal solutions (i.e. control-state admissible pairs) to the control problems and show a result on the variational stability. We apply these result to dynamic and quasi-static nonlinear hemivariational inequalities which model contact problems.

2. Statements of problems

We deal with the following optimal control problem for second order evolution inclusion

$$(\mathrm{P})_1 \qquad \inf\{\, F(u,v) \mid u \in \mathcal{S}_1(v)\,\}$$

where

$$F(u,v) = l(u(T)) + \int_0^T L(t, u(t), u'(t), v(t))\, dt$$

is the cost functional and $\mathcal{S}_1(v)$ is the solution set to the problem

$$(\mathrm{i}) \quad \begin{cases} u''(t) + A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni \\ \qquad\qquad \ni f(t) + G(t)v(t) \ \text{ a.e. } t \\ u(0) = u_0, \quad u'(0) = u_1 \\ v(t) \in U(t) \ \text{ a.e. } t \in (0,T) \end{cases}$$

and v represents a control variable, $G(t)$ is a given family of linear operators. We also consider the following optimal control problem for the first order system:

$$(\mathrm{P})_2 \qquad \inf\{\, F(u,v) \mid u \in \mathcal{S}_2(v)\,\}$$

where $\mathcal{S}_2(v)$ is the solution set to the problem

$$(\mathrm{ii}) \quad \begin{cases} A(t)y'(t) + By(t) + M^*\partial J(t, My(t)) \ni \\ \qquad\qquad \ni f(t) + G(t)u(t) \ \text{ a.e. } t \in (0,T) \\ y(0) = y_0, u(t) \in U(t) \ \text{ a.e. } t \in (0,T). \end{cases}$$

3. Main Results

We are going to obtain the following results

(i) existence of solutions to the state problems (i) and (ii);

(ii) closedness of the graphs of the maps $v \mapsto \mathcal{S}_1(v)$, $v \mapsto \mathcal{S}_2(v)$, where $\mathcal{S}_1(v)$ and $\mathcal{S}_2(v)$ denote the solution sets of the inclusions (i) and (ii), respectively;

(iii) existence of optimal solutions to $(\mathrm{P})_1$ and $(\mathrm{P})_2$;

(iv) asymptotic behavior of a sequence of problems (i) when the acceleration tends to zero and convergence of solutions to (i) to a solution to (ii).

4. Applications to dynamic and quasi-static viscoelastic contact problems

We provide below an example of the dynamic contact problem with nonmonotone nonsmooth subdifferential boundary conditions which can be formulated as the evolution inclusion (1). The corresponding quasi-static model can be formulated analogously.

Let Ω be an open bounded subset of $\mathbb{R}^d$ ($d = 2, 3$ in applications) which represents a deformable viscoelastic body. The boundary Γ of Ω is supposed to be Lipschitz continuous and Γ consists of three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C such that $m(\Gamma_D) > 0$. The body is clamped on Γ_D, the volume forces of density f_0 act in Ω and the surface tractions of density f_1 are applied on Γ_N. The body is in contact with a foundation over a potential contact surface Γ_C. We put $Q = \Omega \times (0,T)$. We denote by $u\colon Q \to \mathbb{R}^d$ the displacement field, by $\sigma\colon Q \to \mathbb{S}^d$ the stress tensor and by $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ the strain tensor, where $\partial_j = \frac{\partial}{\partial x_j}$, i, $j = 1,\dots,d$ and $\mathbb{S}^d$ stands for the space of symmetric matrices of order d.

We assume a linear viscoelastic model with the constitutive law of the Kelvin-Voigt type $\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(u') + b_{ijkl}\varepsilon_{kl}(u)$ in Q, where $\mathcal{C}(t) = \{a_{ijkl}(t)\}$ and $\mathcal{G} = \{b_{ijkl}\}$, i, j, k, $l = 1,\dots,d$ are the viscosity and the elasticity tensors, respectively. Denote by u_ν and u_τ (σ_ν and σ_τ, respectively) the normal and the tangential components of the displacement u (of the stress field σ, respectively) on Γ, i.e. $u_\nu = u \cdot \nu$, $u_\tau = u - u_\nu \nu$ ($\sigma_\nu = (\sigma\nu)\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, respectively), where ν is the outward normal vector to Γ.

Let the functions $j_\nu\colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ and $j_\tau\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz in their last variables and ∂j_ν, ∂j_τ denote their Clarke subdifferentials. The initial displacement and the initial velocity are denoted by u_0 and u_1, respectively. The classical formulation of the mechanical problem is as follows: find a displacement field $u\colon Q \to \mathbb{R}^d$ such that

$$
\begin{aligned}
& u''(t) - \operatorname{Div}\sigma(t) = f_0(t) \ \text{ in } Q \\
& \sigma(t) = \mathcal{C}(t)\varepsilon(u'(t)) + \mathcal{G}\varepsilon(u(t)) \ \text{ in } Q \\
& u(t) = 0 \ \text{ on } \ \Gamma_D \times (0,T) \\
& \sigma(t)\nu = f_1(t) \ \text{ on } \ \Gamma_N \times (0,T) \\
& -\sigma_\nu(t) \in \partial j_\nu(t, u_\nu) \ \text{ on } \ \Gamma_C \times (0,T) \\
& -\sigma_\tau(t) \in \partial j_\tau(t, u_\tau) \ \text{ on } \ \Gamma_C \times (0,T) \\
& u(0) = u_0, \ u'(0) = u_1 \ \text{ in } \ \Omega.
\end{aligned}
$$

The examples of the subdifferential contact boundary conditions can be found in [7, 8, 9, 12].

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